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FIELD MANUAL

FOR

ENGINEERS.

BY

Harvey

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PREFACE.

THE aim in this work has been:

First. To present the subjects of the text in a mathematical and logical order.

Second. To classify all problems presented so as to be easily referred to.

Third. Especial care has been taken to express the resulting formula of every problem in the form requiring the least numerical computation. Many instances of this may be seen by comparing these with similar formulas from other sources.

Fourth. To furnish a large number of useful tables, more complete, more extended, and, where possible, with more elementary and appropriate arguments than other similar tables possess.

Fifth. To treat the general problems of Railway Engineering more extensively than other similar works have done. The growth of the modern railway system during the past half-century has been so rapid that the mathematical needs of the subject have not kept pace with that growth. The technical books on this subject of some two decades ago are entirely inadequate to supply the needs of to-day, and the most recent works are more restricted and elementary than the former. For these reasons the author, in attempting to supply a useful book, seeking utility rather than novelty, has found it necessary to modify some methods and formulas in use, and to introduce new subjects and formulas altogether.

In the following the modified, extended, and extra topics referred to are, for clearness, considered in connection with the general matter of the text.

Table II dispenses with all calculation in laying out curves, as shown in Chapter IV; Table VII greatly simplifies the finding of the tangent and the external of any curve; and Tables XVIII and XIX, and some others to which the remark can apply, are, it is believed, in terms of the proper arguments and in the best form.

In Chapter IV the laws of errors in field-work are demonstrated and illustrated; and the best method of conducting a preliminary survey, introduced by the author a generation ago, is explained.

In Chapter V simple and exact formulas for determining the height of a mountain or other object by the dip of the horizon are substituted in place of the approximate formulas in use.

For the stadiä, as well as for the telemeter, new formulas are found which require no general computation; and the formula for finding the proper elevation on curves, unlike other formulas, involves no large factors.

In addition to a very general treatment of Compound Curves, Chapter VI includes the location of such curves of any number of branches (pp. 143-5), as well as easy and symmetrical formulas for finding their tangents (p. 176).

The reader interested in the philosophy of mathematics will find, it is hoped, an elegant and fruitful illustration of the principles of substitutions in determining general curves to fulfill required conditions on pages 145, 146. This is susceptible of general application. There has also been added a general treatment of the subject of Curves tangent to Curves, including the "Wye Problems"; and also that of Concentric Curves applicable to Parallel Turnouts.

The finding of the angles between the rails at the crossings of curved tracks is also thought to be a valuable addition.

Chapter VII includes a variety of problems in Reversed Curves. The solution of Problems IV, V, and VI were first given to the Senior Class (1869) of the University of Michigan, while the author was in temporary charge.

Chapter VIII treats extensively of Turnouts. The distinction between connecting a straight line and a track, and two tracks, and what is required in each case, is shown on pages 199, 200. The impracticability of certain proposed turnout curves is demonstrated (p. 205), and a variety of methods of laying out turnout curves is shown.

Chapter IX applies entirely to the author's True Transition Curve.

Chapter X shows that the formulas for the computation of earthwork may be abridged one half by supposing that the side slopes are produced in this intersection. A general relation between the "end-area" volume and the "middle-area" volume is shown by means of symbols, and new formulas are given for the volume of a frustrum of a pyramid and the frustrum of a cone. A formula showing the true correction of earthwork for curvature is also deduced, and the best method of computing earthwork tables explained and illustrated.

Chapter XI contains only a brief exposition of the subject of approximate and abridged computations, which it is thought may serve to encourage the shortening of computations.

Chapter XII describes the processes incident to construction and calls attention to two principles that aid very much in "staking out" earthwork.

The logarithmic tables are not reproduced, for the reason that they are but little used and should not be used at all—and most emphatically so in this line of work. It is fair to observe that the space required for such tables is replaced by numerous useful tables, applying directly to the matter in hand, and also to enlarging the subject-matter of the book, thus saving greatly in time and labor. Furthermore, it should be stated that there is no problem in the book requiring a computation more complex than to find the cost of 29 oranges (say), supposing that 17 oranges cost 43 cents. The author must believe that no person—much less an engineer—would think of applying logarithms to the above example; and if so, he could not with any propriety apply them to any problem in the Manual, since the nature of the numerical computation to be made, and not the subject-matter of the problem, furnishes the test of methods.

The author invites criticism, and, should another edition be called for, will make the best use possible of any suggestion that may in good faith be made to him.

The Tables, all but three, were computed expressly for this book, and scrupulous care has been taken, by numerous readings and checks, to make all tables exact to the last figure.

The book, as indicated in a few places, has been in preparation several years and contains matter gathered all along the paths of the author's experience; and is the result of a belief, on his part, of his ability to aid his professional brethren in this direction.

While this delay has not been to the advantage of the author, it has nevertheless given opportunity for due reflection and reconsideration; and therefore, as a work of judgment based on experience, the volume is offered to his brother enquirers. If the book even partially accomplishes the object of the author's aims, he will feel that the days he has devoted to it, though many, have not been spent in vain.

P. H. PHILBRICK.

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FIELD-MANUAL FOR ENGINEERS.

CHAPTER I.

PRELIMINARY OPERATIONS.

1. THE engineering operations preparatory to the construction of a railroad are :

The Reconnoissance ;
The Preliminary Survey or Surveys ; and
The Location.

2. *The Reconnoissance* is a general but incomplete examination of the country through which the proposed road is to pass, made for the purpose of acquiring data upon which surveys may be made and compared, and the best possible route for the road selected.

3. *A Preliminary Survey* consists of the measurement of a line, including its angular deflections ; the elevations of various points upon it, the determinations of the topography along it and near it, for the purpose of furnishing the data from which the line may be definitely located; *or* the survey compared with other surveys, for the purpose of selecting one from which the location may be made.

4. *The Location* consists in placing the line in the exact position in which it is intended to be. This position is called The Location.

5. It is convenient to carry on these operations concurrently.

The main points to consider in the location are the relative cost of grading and bridging, and the relative grades and curvature of the lines. In grading, the character of the soil for stability, in both "cut" and "fill," should be considered.

It is sometimes important to know the relative value of property traversed by different lines; and if the lines are far apart, the probable amount of traffic that the respective lines can command must also be taken into account.

6. It is evident that the best possible location requires the least possible grading, bridging, curvature, etc., taken together, regarding the cost and the expense of operating the road. We can afford, therefore, to increase the curvature, for example, if by so doing we can at the same time decrease the earthwork, and the line is bettered more by the latter than it is damaged by the former.

The field-work of location has for its object to determine the exact position of the selected route on the ground, to establish the grade, to compute the amount of earthwork, decide upon the amount of bridging, etc.

7. A railroad line usually follows the valleys of watercourses or the dividing ridges between watercourses, or crosses valleys and ridges more or less obliquely.

8. The location on dividing ridges is perhaps the simplest of all. In this no bridges and few culverts are required; and the elements governing the location are mainly the amount of earthwork and the curvature of the line. In this case a sketch of the ridge, especially of its prominent features and *governing* points, is made while *walking* over it; and the preliminary line is accurately run, and made into a location, if the route is adopted.

9. The location along the valley of a stream is usually more complex than the former. If the stream is so small that the cost of bridging it would be plainly less than the advantage to the alignment by crossing; or if, on the contrary, the river is so large that the crossing of it is out of the question, the cost of bridging is not considered and the problem of location is reduced, in the main, to that of making the best alignment within the limits of the valley in the one case, or upon one side of the river in the other. The reconnoissance and surveys would be made as already described.

Usually the most favorable ground both for alignment and construction alternates from one side to the other of the stream, and only the results of careful and scientific surveys can tell how many and where the crossings must be in order to make the alignment and grades the best possible, and to secure the most favorable ground. In this case the ground on both sides of the stream must be carefully examined, and if the stream, in consequence of banks or bottom, is not easily crossed at most points, the most suitable crossings must be found. When this is done the engineer will mark out and survey one or more lines so as to fit the chosen crossings and other governing points. A comparison between different lines will point out the best, and generally the one from which the best location can be made.

10. In locating a line across valleys and ridges, the engineer must find the best crossings of the streams, and the gaps or notches in the ridges, and must connect such of the former with such of the latter as will furnish the best line.

11. Sometimes a railroad may occupy either valleys or dividing ridges for the greater part of its length; in which case a choice must be made between the *higher* and the *lower* line.

The higher line will require very much less drainage than the lower line, which is an important advantage; but, on the contrary, probably the curvature and the length of the higher line will exceed that of the lower.

12. It is not to be supposed that, in general, a railroad will follow either a valley or a ridge; or that it will cross valleys and ridges obliquely throughout its entire length; but parts of the line will generally do so, and to these, and therefore to the whole line, the preceding principles will apply.

13. The regular "reaches" in a stream furnish the best crossings. Throughout these, compared with other points, the flow of the current is most uniform; the wash of the bottom, and the caving and the shifting of the banks, are least; while the security of the foundations of bridges and of approaches is greatest. If the bottom of the stream is variable, the best site on some given "reach" must be selected with a view to the kind of foundations suitable to the place. Sharp bends in streams should be studiously avoided.

14. The engineer should freely consult the best maps of the country that he can command, and he should prepare a map on convenient scale, upon which he should copy the principal features of the country, such as streams and lakes, roads and towns, and fill in the details as he progresses. He should also locate on the map the governing points of the route, such as the best crossings of streams, the "gaps" in the ridges, mountain passes, etc. He may then sketch the line on the map.

In a densely wooded country, the making of a thorough reconnoissance is comparatively difficult. In this case it will often be necessary to cross and recross the country many times before a comprehensive knowledge of it can be gained.

15. The one almost indispensable instrument in making a reconnoissance is a pocket-compass. Field-glasses, hand-levels, telemeters, and other instruments are sometimes used, but are rarely needed, and cannot be used to advantage. Those who cannot make a proper reconnoissance without them would better employ some one who can, and turn their attention to other parts of the work.

16. For preliminary surveys the corps of engineers may be constituted as follows: A chief engineer or engineer in charge, an assistant engineer or transitman, a levelman, a rodman, a stakeman, a rear flagman, two chainmen, and one or more axmen according to needs. The head flag should be carried by the head chainman.

17. Since a survey can be made more rapidly with the compass than with the transit, the compass may be used in preliminary work to a limited extent. Owing, however, to the inherent inaccuracies of the compass, a line run with it is generally worthless, except as a guide to a transit preliminary, from which the location may be made. The compass is therefore of very little, if any, use in a comparatively level country, but is useful, if at all, for running the first preliminary line in a hilly region, where several lines must be run. The compass should be light, should be mounted on a Jacob's-staff, and should have a narrow slit in one sight, and a fine platinum wire, or its equivalent, stretched along the center of a wide slit in the other sight. A self-reading rod is best for this work because it saves much time over the sliding rod; and because, too, it enables the levelman to do his own reading. The ax for driving stakes should have a broad head. Stakes should be of a generous length, say from 30 to 36 inches, well driven into the ground, and projecting above the grass and other vegetation, so that the line may be easily followed or recovered, in fields or woods, by the engineer and others. Short stakes occasion the loss of much time, and those shorter than about 30 inches are generally a nuisance, except on streets, well-traveled roads, etc.

18. To make a successful reconnoissance requires a good eye for distances, elevations, etc., and a quick and clear perception of the salient features of the country. One must be able to form a mental picture or image of the country along the proposed line; and of a number of lines crossing and recrossing each other within the limits of his mental map. Thus he may be able to sketch a proposed line, or several proposed lines, which at any time may be tested. The comparison of lines, to test their relative economy, is a question of science, aided by mathematics.

19. It is plain that without skill in making the reconnoissance, the surveys would be at first more or less at random, and the reaching of a location roundabout and expensive. On the other hand, the greatest skill without a *sufficient* knowledge of mathematics and without a knowledge of the principles of engineering bearing upon the question, cannot produce the best location.

A happy combination of the qualities necessary to the successful locating engineer is most assuredly found in comparatively few individuals.

The two main obstacles to contend with in building a railway are grades and curves. Since both affect the cost of construction and the expense of operating the road, such grades and curves as will render the total cost of construction and operation of road a minimum are, as already suggested, the best. Since the resistances due to grades as well as to curves add to other resistances to the movement of trains, and since it is often possible to lessen grades at the expense of curvature, and *vice versa*, it becomes necessary to briefly consider the nature of the resistances that a moving train encounters, with a view to compensation.

The resistances to overcome are, first, the friction of the moving parts of the engine and train; the friction of the wheels on the rails; impacts and oscillations, and the resistance of the air. These resistances vary with the condition of the rolling machinery, the road, and the weather, and are not accurately known. Friction is nearly independent of the speed of the train; but the res ances due to impact increase with the speed, and those due to the atmosphere increase in a still greater ratio.

The sum of these resistances on a level track in fairly good condition and average fair weather is, according to Vose,

$$r = \frac{v^2}{171} + 8, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which r is the resistance in pounds per ton, and v is the velocity in miles per hour. It is not to be supposed that this formula is accurate, though perhaps it is as nearly so as any; and probably the resistances are no greater than the formula indicates. With a velocity of 20 miles per hour the formula gives $r = 10.34$ pounds per ton of the entire weight. Under unfavorable conditions, as a wet, soft, and rough track and high wind, the resistance might be 20 to 40 per cent more, or even greater still.

The second resistance is due to grades of the track. The force necessary to overcome this is such a part of the total weight of the train as the vertical rise of the grade is to its length. Let $r' =$ resistance per ton, and $h =$ the ascent per 100 feet. Then $r' = \frac{h}{100} \times 2000$ pounds per ton. For a 1% grade (52.8 feet per mile) $h = 1$ and $r' = 20$ pounds per ton, the force necessary to overcome the load. A ½% grade would of course create a resistance of 10 pounds per ton, or approximately the same as the resistance caused by friction, impact, resistance of the atmosphere, etc., at a speed of 20 miles per hour, as stated above.

The third resistance is due to the curves of the track. This resistance is not accurately known. On American roads with American rolling stock it is probably about one half a pound per ton for each degree of curvature. Letting r'' represent the resistance per ton per station, and D the degree of curvature, the resistance per station is $r'' = .5D$. We observe that the resistance of a 1° curve is but the $\frac{1}{40}$ part of that due to a 1% grade, or equivalent to $\frac{52.8}{40} = 1.32$ feet per mile. The resistance of a 5° curve is equivalent to 6.6 feet per mile, and that of a 10° curve to 13.2 feet per mile, etc. The resistance offered to a train in moving through 100 feet of a 1° curve is one-half pound per ton moved 100 feet, which is equivalent to the total load lifted $\frac{\frac{1}{2}}{2000} \times 100 = \frac{1}{40} = .025$ ft.; and since the resistance varies as the product of the degree of curvature and length of the curve, or as the total curvature, the resistance on any curve for each degree of change of direction is equal to the lifting of the train $\frac{1}{40}$ of a foot. Since an engine can haul over a road only what it can haul over the most unfavorable part, easy curves are necessary where heavy grades must occur, and *vice versa* ; the object being, of course, to keep at all points the sum of the resistances due to grades and curves below the allowable maximum. In equating for grades and curves, however, the ruling element is largely the total ascent and not the slope merely, as is too often considered to be the criterion. No more power is required to make an ascent of 20 feet, for example, on a 2% grade 1000 feet long than is required on a ½% grade 4000 feet long.

To compare grades simply, that is, the *rate* of ascent or descent especially on short grades, as is usually done, is meaningless at least—it is absurd.

On light grades no allowance usually need be made for curvature, since the momentum of the train would carry it over the ascent even without aid from the engine. Grades situated near stations, however, or other points where a train must stop or "slow up" are greater obstacles than in other situations to the movement of trains, and upon such grades an allowance for curvature, as above shown, should be made.

Every locating engineer is presumed to make himself familiar with the principles involved in locations before making them; and to aid in this the reader is referred to Vose and others, and especially to the elaborate treatise of Wellington on "Railway Location." Wellington gives in Table 118 the "Total Energy, or Potential Lift, in Vertical Feet in Trains moving at Various Velocities," and discusses at length all subjects connected with location.

CHAPTER II.

ADJUSTMENTS, USES, AND CARE OF INSTRUMENTS.

THE TRANSIT.

The following are the usual adjustments of the transit:

A. *To Adjust the Levels* (that is, to place the levels in a plane perpendicular to the upright axis).—Set up the instrument upon its tripod as nearly level as may be; unclamp the plates and bring the two levels above and on a line with the two pairs of leveling-screws. Then by means of one pair of screws bring the bubble of the level above them to the middle of the opening. Without moving the instrument bring the other bubble to the middle in the same way. Since in moving one pair of screws very far the other pair is liable to become cramped and the corresponding bubble somewhat disturbed, it is advisable to bring the bubbles in succession near the middle, repeating if necessary, and ending by bringing them exactly to the middle.

When both bubbles are in place turn the instrument through about 180°; if the bubbles are now in position they need no correction; but if not, turn the small nuts at the end of the levels until the bubbles are moved over one half of the error. Then bring the bubbles again to the middle by the leveling-screws and repeat the operation just described until the bubbles will remain in the middle during a complete revolution of the telescope. This shows the adjustment to be completed.

B. *To Set the Cross-wires Vertical and Horizontal.*—Level the instrument. Move the telescope upward or downward and note whether the vertical wire traverses some fixed point or not. If not, loosen the four cross-wire screws and, by the pressure of the hands on their head outside the tube, move the cross-wire ring around, what seems to be sufficient, and repeat the operation, if necessary, until the vertical wire will traverse some fixed point.

The cross-wire screws are near the screws of the centering-ring of the eyepiece (the heads of which are usually covered by an outside ring), but between them and the axis of the telescope.

C. *To Make the Line of Collimation Perpendicular to the Axis of the Wyes, so that it will revolve in a plane.*—Set up the transit at some point *O* near the middle of a level piece of ground and level it carefully. Let *AOB* be a straight line, *AO* and *BO* being nearly equal. Direct the line of sight to *A*, clamp the instrument and revolve the telescope, and if the line of collimation is *not* perpendicular to the axis, the line of sight will determine a point *C*, say, on one side of *B*. To test the matter, loosen the upper clamp and turn the vernier plate almost or quite half-way around, so that the line of sight will again be on *A*, and clamp the plates. Again revolve the telescope and note the point *D*, suppose, thus determined. If the point *C* coincides with *B*, *D* will also coincide with *B* and the line of collimation is in adjustment. If, however, *C* is on one side of *B*, as shown, *D* will be equally far on the other side, showing the line of collimation out of adjustment.

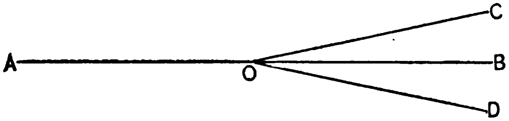


FIG. 1.

To correct the error, use the two capstan-head screws on the side of the telescope, to move the ring to which the wires are fastened laterally, and with it, of course, the intersection of the wires.

Having moved the vertical wire until by estimation one fourth of the space *DC* is passed over, return to *A*, clamp and revolve the telescope, and if the correction has been carefully made the line of sight will now cut the point *B*. It will, however, generally be necessary to repeat the operation, the adjustment not being perfected at first.

Remember that the eyepiece inverts the position of the wires, and therefore in moving the ring the operator must proceed so as seemingly to increase the error.

The wires may now be brought into the center of the field of view by moving the screws of the centering-ring of the eyepiece, which are slackened and tightened in pairs, the movement being now direct until the wires are seen in their proper position.

It is proper to observe that the position of the line of collimation depends solely upon that of the objective, so that the eyepiece may be moved in any direction, or replaced by another without at all deranging or in any way affecting the adjustment of the wires.

D. *To Adjust the Standards to the Same Height so that the Line of Collimation will Revolve in a Vertical Plane.*—Set the transit as close as convenient to the base of a lofty spire, or other high object; level it carefully and clamp it, and direct the telescope to the top of the spire or other elevated and well-defined point, as *A* in the figure. *B* is in the same vertical plane with *A* and the instrument, that is, with the line of sight.

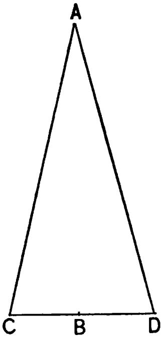


FIG. 2.

Turn down the telescope to some good point on the ground, either found or marked, as *C*. Unclamp the plates or spindle, revolve the telescope and turn it half-way around, or far enough to again sight to the high point. Again clamp and turn down the telescope to some point *D* opposite *C*.

In setting *C* the left standard must have been too high, and in setting *D* the same standard (now the right by revolving the telescope) shows to be equally high, the errors *BC* and *BD* being equal. Correct the error by raising or lowering the sliding-piece at one end of the axis by means of the screw beneath and those above, so that when the telescope is directed to *A* and lowered it will cut *B* half-way between *C* and *D*. If the instrument is in adjustment it will, of course, cut *B* instead of *C* and *D* in the first two trials.

E. *To Adjust the Vertical Circle.*—Set up the instrument firmly, level it carefully, bring into line the zeros of the circle and vernier, and with the telescope find some well-defined point, from two to five hundred feet distant, which is cut by the horizontal wire. Turn the instrument half-way around, revolve the telescope, fix the wire upon the same point as before, and observe if the zeros are again in line. If not, loosen the capstan-head screws which fasten the vernier, and move the zero of the vernier over half of the error. Bring the zeros again into coincidence and proceed as before, as many times as necessary, until the error is entirely corrected.

F. *To Adjust the Level on Telescope.*—First level carefully and clamp the telescope approximately horizontal by the eye.

Then, having the line of collimation previously adjusted, drive a stake, say two or three hundred feet away, and note the height cut by the horizontal wire upon a staff set on top of the stake.

Fix another stake in the opposite direction and at the same distance from the instrument, and without disturbing the telescope turn the instrument upon its spindle, set the staff upon the stake, and drive the stake into the ground until the reading is the same as on the first stake. The tops of the two stakes are equally high however much the telescope may be out of level.

Now set the instrument some twenty or thirty feet from one of the stakes and on the prolongation of the line joining the two. Level the instrument, clamp the telescope as nearly horizontal as may be, and note the readings of the staff on the two stakes. If they agree, the telescope is level. If they do not agree, then with the tangent-screw move the wire over fully the whole error, as shown at the distant stake; repeat the operation just described as many times as may be necessary, so that the wire will give the same reading at both stakes, showing that the telescope is truly horizontal. Taking care not to disturb the position of the telescope, bring the bubble into the middle by the little leveling-nuts at the ends of the tube, which will render the adjustment complete.

The above are all the adjustments usually required at the hands of the engineer; for any others see the *Manual* of W. & L. E. Gurley.

USE AND CARE OF TRANSIT.

The instrument should be set up firmly with the plates nearly level, and thus save much time in turning the leveling-screws, besides the worse than useless wear upon them.

To do this: Set up the transit approximately over the desired point. Hold firmly the leg of the tripod on the left with the left hand, and move the foot of the leg on the right with the right hand in any direction, so as to bring the lower plate of the tripod head approximately horizontal, determined by placing the *eyes* in the plane of its upper surface and sighting as nearly as may be to the distant horizon. It will do no harm to bend the body the trifle necessary to do this. This sight is so long that, according to the principle of the "division of errors," the plate can be readil set very nearly horizontal in almost any position of the instrument.

Now place the left hip against one leg; grasp the opposite le with the left hand, and the leg on the right with the right hand and keeping the relative position of the legs unaltered, move th instrument laterally over the desired point and lower it to th ground.

Of course the two movements of tilting and sliding may be don together and in much less time than required to describe th movement.

TO MEASURE THE ANGLE BETWEEN TWO LINES OR OBJECTS.

Level the instrument carefully; bring the zeros of the vernier and limb together by means of the upper clamp and tangent-screw and direct the telescope toward one of the objects by means o the lower clamp and tangent-screw. Upon loosening the uppe clamp and directing the telescope toward the second object, th angle desired is then shown upon the limb.

Before making an observation with the telescope, the eyepiece should be moved in or out until the cross-wires may be distinctly seen. This may be accomplished with the greatest precision by directing the telescope toward some white object. The sky will serve very well for this. The objective is then adjusted by moving it in or out until the object is seen clear and well defined, and the cross-wires appear as if fastened to its surface.

Water and dust are very destructive to instruments; indeed, the most destructive of all agents is *dust*. An instrument may be badly bruised, bent, or broken, by a fall or otherwise, and yet it may be repaired. If, however, it is allowed to stand on its tripod in a dusty office—and all offices are dusty—its life will be short. The author has known of instruments so choice that their owners would allow no one to handle them lest they might become slightly soiled, or some other mishap befall them; and yet in a few months they were ruined, in the way pointed out. A minute quantity of dust on the sockets will probably cause them to grind; and more of it will increase the grinding so that the instrument will soon become useless.

Whenever not in actual use the eyepiece should be covered by the lid, and the object-glass by the cap, to protect them from dust, moisture, rain, etc.

When an instrument is exposed to the hot rays of the sun for some time its parts are subjected to very unequal expansion, which throws the instrument more or less out of adjustment, thus to some extent vitiating the work and damaging the finer parts of the instrument. To prevent this it should be shielded by an umbrella or screen of some kind supported on top of a high stake; or, if nothing better is at hand, a cloth should be placed over the telescope.

When an instrument is in the office or in transport it should be clamped and placed in a stable position, upon proper supports, in a tight box, well cushioned if possible.

If the box cover does not fit upon the box closely it can be remedied by sticking suitable cloth on the top edge of the box or on the under side of the lid.

When handling an instrument it should be supported by placing the hand under the lower plate.

Tangent and micrometer screws should be used equally on all portions of their length.

Keep the tripod legs tight enough on the tripod head to secure stability by tightening the nuts on the bolts when necessary. Also tighten the shoes of the tripod if they become loose. Neglect of these things sometimes prevents an engineer from keeping his telescope on a point, and causes him to run a zigzag line without knowing the cause of it. Secure the instrument well to the tripod head before using it; and bring all four leveling-screws to a bearing and cover the instrument with an oilcloth hood before carrying it. Use a fine camel-hair brush or a piece of old and soft linen to clean the glasses of the telescope.

Dust, moisture, perspiration, etc., will sometimes cause a film to form on the lenses of a telescope which may greatly impair the sight through it.

To remove the film, the lenses, after being carefully brushed, should be gently wiped with a piece of chamois-skin moistened with alcohol, and the lenses must be wiped dry by using fresh portions of the skin on separate parts of the lens.

To remove dampness in the main tube of the telescope, take out the eyepiece, cover the open end with cloth, and leave the instrument in a dry room for some time.

The centers of an instrument should always be lubricated with fine watch-oil only, and after a careful cleaning. First wipe off all old grit and oil before applying fresh oil.

If dust settles on the cross-wires, unscrew the eyepiece and the object-glass, and gently blow through the telescope tube; cover up both ends and wait a few minutes before replacing the eyepiece and object-glass.

Be sure to bring the object-glass cell to a firm bearing against its shoulder, and then examine the adjustment of the lines of collimation.

To clean the threads of a leveling or tangent screw use a stiff tooth-brush to remove the dust, then apply a little oil and turn the screws in and out with alternate brushing to remove dust and oil, until it moves freely and smoothly. Use such a brush to clean the object-slide. No screws should be strained more than necessary to insure a firm bearing, and this applies with special force to the cross-wire screws.

All straining of such screws beyond this impairs the accuracy of the instrument and the reliability of adjustment.

These remarks in reference to the *care* of the transit apply substantially, of course, to all instruments.

If obliged to work with an instrument of faulty graduation, it is best to read each angle on different parts of the circle, and take the mean value. If we take the mean result obtained by both verniers, we eliminate the errors due to eccentricity of the vertical axis, and also reduce the errors of graduation.

For greater accuracy clamp the vernier-plate to zero and read the angle by both verniers. Then keep the vernier-plate clamped, point the telescope to the first object and proceed as before, any number of times. Then read the verniers, adding 360° for each complete revolution which has been made, and divide this sum by the number of times the angle was read. The quotient is the required angle.

The cross-wires can be illuminated easily by placing a piece of white cardboard, with a hole through it for the line of sight, in front of the telescope, and in an oblique position, so as to reflect into the telescope the rays of a lamp or of a lantern placed back of the object-glass and near the telescope.

The Level.

The principal adjustments of the level consist in the following:

1. Bringing the cross-wires into the optical axis of the tele-

scope and consequently parallel to the line of bearings of the wye rings.

2. Causing the line of bearings to be parallel to the plane of the level.

3. Making either of these lines, and therefore all of them, parallel to the bar and consequently perpendicular to the axis of the instrument.

1. To adjust the line of collimation, that is, to bring the cross-wires into the optical axis, so that their point of intersection will remain on any given point during an entire revolution of the telescope.

Set the tripod firmly, remove the wye-pins from the clips, so as to allow the telescope to turn freely; clamp the instrument to the leveling-head, and by the leveling and tangent screws bring either of the wires upon the clearly defined edge of some object. Then carefully rotate the telescope half-way around.

If the wire does not coincide with the line observed, bring it half-way back by means of the capstan-head screws at right angles with it, always remembering the inverting property of the eyepiece. Then by the leveling and tangent screws bring it again upon the "edge," and repeat the above operation if necessary, and continue to do so until the telescope may be rotated without changing the position of the wires.

If both wires are much out of position, it will be well to approximately adjust the wires alternately before attempting to make the adjustment of either one complete, since an error in one somewhat affects the other.

It may be advisable to center the eyepiece. To do so unscrew the covering of the eyepiece centering-screws, and move each pair in succession, with a screw-driver, until the wires are brought into the center of the field of view.

The inverting property of the *eyepiece* does not affect this operation, and the screws are moved directly. To test the centering, rotate the telescope, and if an object observed appears to change position the centering is not perfect.

In all telescopes the line of collimation is determined by the cross-wires and objective, and is not affected in any way by the eyepiece.

2. To make the line of bearings parallel to the bubble-tube.

so as to insure that it is horizontal, when the bubble is in the center. This adjustment embraces two parts:

First, to bring the center-line of the bubble and the line of bearings in the same plane.

Second, to make these lines parallel.

To effect the first: Clamp the level and bring the bubble to the center by the parallel-plate screws. Now rotate the telescope in the wyes 20°, more or less. If the bubble runs toward the end, it shows that the center-line of the bubble and the axis of the telescope are not in the same plane; in other words, the bubble-tube lies crosswise of the telescope.

To correct the error, bring the bubble by estimation half-way back, by the capstan-head screws, which are set in either side of the level-holder.

Again bring the level-tube under the telescope, the bubble to the middle, etc., repeating the operation just described until the bubble will keep its position, when the telescope is revolved.

For the second part: Bring the bubble to the middle of the tube by the leveling-screws, and take the telescope out of the wyes carefully and turn it end for end.

If the bubble runs toward either end, lower that end, or raise the other by turning the adjusting-nuts on one end of the level until by estimation half the correction is made. Again bring the bubble to the middle by the leveling-screws, and repeat the whole operation just described until the reversion can be made without causing any change in the bubble.

3. Having made the previous adjustments, it remains to make the level-bubble (and therefore the line of bearings and line of collimation) parallel to the bar, and therefore perpendicular to the vertical axis of the level, so that the bubble will remain in the middle during an entire revolution of the telescope.

Place the level over a pair of leveling-screws, and by means of them bring the bubble to the middle. Turn the instrument half-way around horizontally. If the bubble runs toward either end, bring it half-way back by either pair of nuts, at the ends of the bar. Then bring the bubble to the middle again, by the leveling-screws, etc., repeating the operation just described, until the bubble will remain in the middle of the tube when the instrument is revolved.

In making this adjustment it is best to use the opposite pairs of leveling-screws alternately, thus bringing the upper parallel plate of the tripod head into a position as nearly horizontal as possible, so that the error caused by not revolving the instrument precisely 180° may be the least possible.

This adjustment is for convenience and not for accuracy in any appreciable degree.

Now turn the telescope in the wyes until the pin on the clip of the wye will rest in the little recess in the ring to which it is fitted.

Apply the horizontal wire to any level line, and in case it does not coincide with it, loosen two cross-wire screws at right angles to each other, and by their heads outside turn the cross-wire ring until the horizontal wire coincides with the level line.

The line of collimation must then be adjusted again. In readjusting the line of collimation none of the lines referred to in making the adjustment is disturbed, and the adjustments are complete.

ADJUSTING BY THE "PEG" METHOD.

By this method the main adjustments are effected at once.

Drive two pegs several hundred feet apart, and set the instrument midway between them. Read the rod on each, keeping the bubble in exactly the same position, preferably at the center. The difference of the readings is the difference of the heights of the pegs, no matter how much or in what way the level may be out of adjustment. Then set over either peg and measure the height of cross-wires above top of peg.

The difference of heights of pegs, added to this, or taken from it, according as the instrument is over the higher or lower peg, gives the height of the cross-wires above the other peg. Set the rod on that peg, and bring the horizontal wire to that height on the rod by the leveling-screws, keeping them at a bearing. Then bring the bubble to the center by raising or lowering one end of the level-tube.

The first part of the second adjustment,—namely, to bring the level-bubble and line of collimation in the same plane,—also the third adjustment, should be made as heretofore.

USE OF THE LEVEL.

The instrument should be set up firmly, with the top of the tripod as nearly level as may be, so as to save time in leveling and the wear of the screws, etc.

The setting up of the level is, of course, similar to that of the transit already described.

The bubble should then be brought over each pair of leveling-screws successively, and leveled in each position.

Bring the wire precisely in focus by the eyepiece, and the object distinctly in view by the objective, so as to avoid all "traveling of the wires" or parallax.

It is best, where practicable, to take approximately equal fore and back sights, so as to eliminate any error due to a lack of perfect adjustment, which is difficult to secure.

For precise reading the rod should not be over 400 or 500 feet from the instrument.

If the socket of the instrument sticks in the leveling-head so as to be difficult to remove, be sure that the instrument is unclamped and the leveling-screws are free. Then place the palms of the hands under the wye-nuts under each end of the bar, and give a sudden upward blow to the bar, and take care also to grasp it the moment it is free.

TO ADJUST THE COMPASS.

The Levels.—First bring the level-bubbles into the middle by the pressure of the hands on different parts of the plate; then turn the compass half-way around. If either bubble runs toward one end of its tube, it indicates that that end is too high.

Lower it by loosening the screw under the lower end, and tightening the one under the higher end, until the error is, by estimation, half removed. Level the plate again, and repeat the operation until the bubbles will remain in the middle during an entire revolution of the compass.

The Sight-vanes.—The sights may next be tested by observing through the slits a fine hair or thread made exactly vertical by a plummet.

If either slit does not coincide in direction with the hair or thread, it must be made to do so by filing its under surface on the higher side.

The Needle.—Having the eye nearly in the same plane with the graduated rim of the compass circle, observe whether or not the ends of the needle cut opposite points on the rim. If not, bend the center-pin (by means of a small wrench) about an eighth of an inch below the point of the pin, so as to make the ends cut opposite points on the rim.

The needle now may be supposed to occupy the position *NpS*, the pivot *p* not being in the center *O*. Now keeping the needle in the same position, turn the compass half-way around. The needle will now occupy the position *N′p′S′*. Correct half the error by bending (that is, ***straightening***) the needle, making it cut points half-way between its former positions, so that it occupies the position *Ap′B* and is straight; and correct the other half by bending the pin, placing the point of the pin at *O* and giving the needle the position *NOS*.

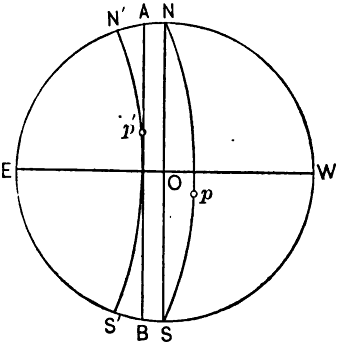


FIG. 3.

The operation should be repeated until perfect reversion is secured in the first position.

Then try the needle on another quarter of the circle, and if an error is manifested, correct the center-pin only, the needle being already straightened by the previous operation.

Do the same on other quarters of the circle until the needle will reverse in any position.

TO USE THE COMPASS.

In using the compass keep the south end toward the person, and read the bearings from the north end of the needle.

Mark every station or point at which the compass is set, so that it may be easily found for verification or a resurvey. It is much more important to have the compass level laterally, or crosswise of the sights, than in their direction; since if it is not so, on looking up or down hill through the lower part of one sight and the upper part of the other the line of sight will not be parallel to the N. and S. or zero line on the compass, and an incorrect bearing will be obtained.

A continuous line thus run, to say nothing of other imperfections of the compass, would be a zigzag line probably very much in error.

The compass cannot be leveled by the needle, for the dip of the needle is continually varying. If the needle touches the glass when the compass is leveled, balance it by sliding the coil of wire along it.

The vibrations of the needle may be checked by gently raising it off the pivot, so as to touch the glass, and letting it down again, by the screw on the under side of the box.

The compass should be smartly tapped after the needle has settled, to destroy the effect of any adhesion to the pivot or friction of dust upon it.

The glass sometimes becomes charged with electricity by carrying it against clothing, or wiping it, etc., so that it attracts the needle to its under surface, preventing its free movement.

The difficulty may be remedied by breathing on the glass, or touching it in different places with the moistened finger.

Of course the chain and all other metals should be kept away from the needle.

CHAPTER III.

PLANE TRIGONOMETRY.

1. Plane Trigonometry treats of the relations of the sides and angles, and of the solution of plane triangles.

2. Some French writers divide the right angle into 100 degrees, the degree into 100 minutes, the minute into 100 seconds, etc. This *centesimal* system, in which reductions are made by simply moving the decimal point, is altogether preferable to our *sexagesimal* system.

3. Two angles whose sum is equal to 90° are complementary.

Two angles whose sum is 180° are supplementary.

4. Let us consider a series of right triangles ABC, $AB'C'$, etc., having the common angle A. The triangles are equiangular and therefore similar, and we have

$$\frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''}; \quad (1)$$

$$\frac{BC}{AC} = \frac{B'C'}{AC'} = \frac{B''C''}{AC''}; \quad (2)$$

$$\frac{AB}{AC} = \frac{AB'}{AC'} = \frac{AB''}{AC''}. \quad (3)$$

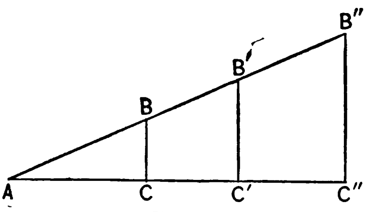


FIG. 4.

Thus it appears that the ratios of the sides are the same in all right triangles having the same acute angles; and therefore if these ratios are known in any one of these triangles, they will be known in all of them.

As any triangle may be divided into two right triangles, it is evident that the solution of oblique triangles may be made to depend upon the solution of right triangles.

The above *ratios*, depending upon the angle alone and not at all upon the absolute lengths of the sides, may be considered as indices of the angle, and have received special names, which we will explain.

5. Let us represent the sides and angles of a triangle in the usual way, shown in Fig. 5. The side opposite an angle, divided by the hypothenuse, is called the sine of that angle. Thus

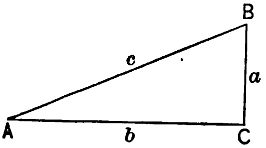


FIG. 5.

$$\frac{a}{c} = \sin A, \quad \text{and} \quad \frac{b}{c} = \sin B.$$

The side opposite an angle, divided by the adjacent side, is called the tangent of that angle. Thus

$$\frac{a}{b} = \tan A; \qquad \frac{b}{a} = \tan B.$$

The hypothenuse, divided by a side adjacent to an angle, is called the secant of that angle. Thus

$$\frac{c}{b} = \sec A; \qquad \frac{c}{a} = \sec B.$$

6. The terms cosine, cotangent, and cosecant are convenient abbreviations for the "sine of the complement," "tangent of the complement," and "secant of the complement," respectively. The reader must not suppose that there is such a thing or entity as cosine, cotangent, or cosecant of an angle.

Since the acute angles of a right-angled triangle are complementary, that is, $A = 90° - B$ and $B = 90° - A$, it follows that

cos A *means* the sine of $(90° - A)$, or sin B;
cot A *means* the tangent of $(90° - A)$, or tan B;
cosec A *means* the secant of $(90° - A)$, or sec B, etc.

Hence

$$\left.\begin{array}{ll} \sin A = \cos B = \dfrac{a}{c}; & \cos A = \sin B = \dfrac{b}{c}; \\ \tan A = \cot B = \dfrac{a}{b}; & \cot A = \tan B = \dfrac{b}{a}; \\ \sec A = \operatorname{cosec} B = \dfrac{c}{b}; & \operatorname{cosec} A = \sec B = \dfrac{c}{a}. \end{array}\right\} \quad . \quad (4)$$

Sin $A = \cos B$ is really an identical equation, since $\cos B$ *is* the sine $(90° - B) = \sin A$; and so is $\tan A = \cot B$, etc. Furthermore,

$$\text{vers } A = \frac{c - b}{c} = 1 - \cos A.$$

$$\text{exsec } A = \frac{c - b}{b} = \sec A - 1 = \frac{1}{\cos A} - 1 = \frac{1 - \cos A}{\cos A} = \frac{\text{vers } A}{\cos A}.$$

7. Plane trigonometry, applied to plane triangles, is but the application of the *one* well-known proposition in geometry: "Equiangular triangles have their homologous sides proportional and are similar."

Comparing Figs. 5 and 6, we observe that the above ratios $\frac{a}{c}$, $\frac{a}{b}$, etc., change as the angles A and B change. These ratios have been computed, however, for all values of the angles, differing by single minutes or less, and placed in tables under the corresponding headings, sine, tangent, etc., and opposite the corresponding angles.

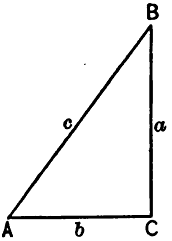


FIG. 6.

8. From the above equations we observe that

$$\text{cosec } A = \frac{1}{\sin A}; \quad \dots\dots \quad (5)$$

$$\cos A = \frac{1}{\sec A}; \quad \dots\dots \quad (6)$$

$$\cot A = \frac{1}{\tan A}. \quad \dots\dots \quad (7)$$

Hence

$$\sin A \text{ cosec } A = \sec A \cos A = \tan A \cot A = 1. \quad . \quad (8)$$

We also have, by division,

$$\frac{\sin A}{\cos A} = \frac{a}{b} = \tan A; \quad \dots\dots \quad (9)$$

$$\frac{\cos A}{\sin A} = \frac{b}{a} = \cot A. \quad \dots\dots \quad (10)$$

Again,

$$\sin^2 A + \cos^2 A = \frac{a^2 + b^2}{c^2} = \frac{c^2}{c^2} = 1. \quad . \quad . \quad . \quad (11)$$

Again,

$$\sec^2 A = \frac{c^2}{b^2} = \frac{a^2 + b^2}{b^2} = 1 + \frac{a^2}{b^2} = 1 + \tan^2 A,$$

and

$$\sec A = \sqrt{1 + \tan^2 A}; \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\operatorname{cosec}^2 A = \sec^2 B = \frac{c^2}{a^2} = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2} = 1 + \cot^2 A,$$

and

$$\operatorname{cosec} A = \sqrt{1 + \cot^2 A}. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (9),

$$\sin A = \cos A \tan A = \frac{\tan A}{\sec A} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}. \quad . \quad (14)$$

From (10),

$$\cos A = \sin A \cot A = \frac{\cot A}{\operatorname{cosec} A} = \frac{\cot A}{\sqrt{1 + \cot^2 A}}. \quad (15)$$

If one of the above functions of the angle A is given, all the others may easily be found. For example, if $\sin A$ is given, we have, from (11),

$$\cos A = \sqrt{1 - \sin^2 A}.$$

Then

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}; \quad . \quad . \quad . \quad . \quad (16)$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{\sqrt{1 - \sin^2 A}}{\sin A}; \quad . \quad . \quad . \quad (17)$$

$$\sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1 - \sin^2 A}}; \quad . \quad . \quad . \quad (18)$$

$$\operatorname{cosec} A = \frac{1}{\sin A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If cos A is given, we have, from (11),

$$\sin A = \sqrt{1 - \cos^2 A}.$$

Then tan A, etc., as above.

Similarly when other functions of the angle are given.

9. The sine and cosine of two angles being given, to find the sine and cosine of their sum, and the sine and cosine of the difference of their angles.

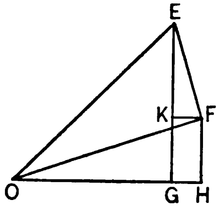


FIG. 7.

In Figs. 7 and 8 let $HOF = A$, and $FOE = B$; then, in Fig. 7, $EOG = A + B$, and, in Fig. 8, $EOG = A - B$. From any point F in OF draw FE perpendicular to OF; also EG and FH perpendicular to OH, and FK perpendicular to EG. Now the three sides of the triangle FEK are perpendicular to the three sides of the triangle FOH, and therefore $FEK = FOH$.

Now, in Fig. 7,

$$\sin(A + B) = \frac{EG}{EO} = \frac{FH + EK}{EO} = \frac{FH}{EO} + \frac{EK}{EO}.$$

But

$$\frac{FH}{EO} = \frac{FH}{FO} \cdot \frac{FO}{EO} = \sin A \cos B,$$

and

$$\frac{EK}{EO} = \frac{EK}{EF} \cdot \frac{EF}{EO} = \cos A \sin B.$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B. \quad . \quad . \quad . \quad (20)$$

Again,

$$\cos(A + B) = \frac{OG}{OE} = \frac{OH - KF}{OE} = \frac{OH}{OE} - \frac{KF}{OE}$$

$$= \frac{OH}{OF} \cdot \frac{OF}{OE} - \frac{KF}{EF} \cdot \frac{EF}{OE} = \cos A \cos B - \sin A \sin B. \quad (21)$$

Then, in Fig. 8,

$$\sin(A-B)=\frac{EG}{EO}=\frac{HF-KE}{EO}=\frac{HF}{OF}\cdot\frac{OF}{EO}-\frac{KE}{FE}\cdot\frac{FE}{EO}$$

$$=\sin A\cos B-\cos A\sin B;\quad\ldots\ldots\quad(22)$$

$$\cos(A-B)=\frac{OG}{OE}=\frac{OH+FK}{OE}=\frac{OH}{OF}\cdot\frac{OF}{EO}+\frac{FK}{FE}\cdot\frac{FE}{OE}$$

$$=\cos A\cos B+\sin A\sin B.\quad\ldots\ldots\quad(23)$$

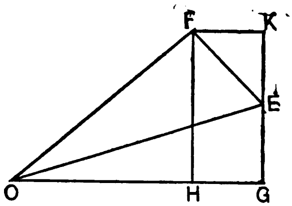


FIG. 8.

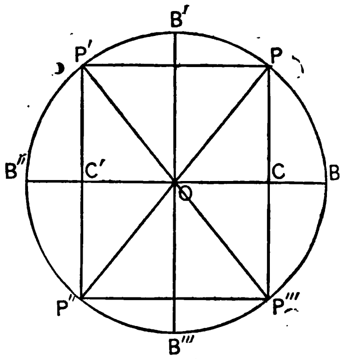


FIG. 9.

In (20), make $B=A$ and get

$$\sin 2A=2\sin A\cos A.\quad(24)$$

In (21), make $B=A$ and get

$$\cos 2A=\cos^2 A-\sin^2 A$$

$$=(1-\sin^2 A)-\sin^2 A=1-2\sin^2 A.\quad\ldots\quad(25)$$

$$=\cos^2 A-(1-\cos^2 A)=2\cos^2 A-1.\quad\ldots\quad(26)$$

(20) and (21) give

$$\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\times\dfrac{\sin B}{\cos B}},$$

or

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.\quad\ldots\ldots\quad(27)$$

Similarly, from (22) and (23),

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.\quad\ldots\ldots\quad(28)$$

We note some special values of the trigonometric functions. See Fig. 9.

Let COP represent any triangle ; angle $COP = O$. Let $O = 0$. Then $CP = 0$, PO and BO coincide and are equal.

Hence

$$\sin 0 = \frac{0}{BO} = 0, \quad \tan 0 = \frac{0}{BO} = 0, \quad \sec 0 = \frac{BO}{BO} = 1.$$

Let $O = 90°$. Then PC and PO coincide with $B'O$, and $CO = 0$.

Hence

$$\sin 90° = \frac{B'O}{B'O} = 1, \quad \tan 90° = \frac{B'O}{0} = \infty, \quad \sec 90° = \frac{B'O}{0} = \infty.$$

Let $O = 180°$. Then PO and CO coincide with $B''O$ and $PC = 0$.

Then

$$\sin 180° = \frac{0}{B''O} = 0, \quad \tan 180° = \frac{0}{B''O} = 0, \quad \sec 180° = \frac{B''O}{B''O} = 1.$$

Let $O = 270°$. Then PO and PC coincide with $B'''O$ and $CO = 0$.

Hence

$$\sin 270° = \frac{B'''O}{B'''O} = 1, \quad \tan 270° = \frac{B'''O}{0} = \infty, \quad \sec 270° = \frac{B'''O}{0} = \infty.$$

The values of the functions of 360° are the same as those of 0°. We need not consider angles greater than 180°.

SOLUTION OF PLANE RIGHT TRIANGLES.

16. In order to solve a plane right triangle it is only necessary to select from equations (4) an equation containing the *two given parts* aside from the right angle and the *part sought*. By transposing, if necessary, so as to express the latter in terms of the former, it becomes known. There are two cases :

CASE I.—A side and an angle given. Given A and c. See Fig. 5.

Example.—Let $A = 35° \, 23'$, and $c = 874.8$. We have

$$B = 54° \, 37'.$$

Also table of sines and cosines gives

$$\sin 35° \, 23' = .57904, \quad \text{and} \quad \cos 35° \, 23' = .81530.$$

Hence

$$a = c \sin A = 874.8 \times .57904 = 506.54;$$
$$b = c \cos A = 874.8 \times .81530 = 713.22.$$

CASE II.—Given two sides.

Example.—Let $a = 184.3$, and $c = 246$. We have

$$\sin A = \frac{a}{c} = .74919.$$

We find in the table

$$\sin 48^\circ\ 31' = .74915, \quad \therefore\ A = 48^\circ\ 31' \text{ to the nearest minute;}$$

$$B = 90^\circ - A = 41^\circ\ 29'; \quad b = c \cos A = 246 \times .6624 = 162.95,$$

or

$$b = \sqrt{c^2 - a^2} = 162.95.$$

TABLE FOR SOLUTION OF RIGHT TRIANGLES.

Given.	Required.	Formulas.
1. a, b	A, B, c	$\tan A = \frac{a}{b}$, $B = 90^\circ - A$, $c = b \sec A = a \sec B$.
2. a, c	A, B, b	$\sin A = \frac{a}{c}$, $B = 90^\circ - A$, $b = c \cos A = a \tan B$.
3. A, a	B, b, c	$B = 90^\circ - A$, $b = a \tan B$, $c = a \sec B = b \sec A$.
4. A, b	B, a, c	$B = 90^\circ - A$, $a = b \tan A$, $c = a \sec B = b \sec A$.
5. A, c	B, a, b	$B = 90^\circ - A$, $a = c \sin A$, $b = a \tan B = c \cos A$.

If, in the second case above, b is given in place of a, then a and b as well as A and B change places in the formulas. However, $A = 90^\circ - B$ is the same as $B = 90^\circ - A$.

17. We will now deduce formulas for the solution of oblique triangles.

Draw BD in Fig. 10 perpendicular to AC. Then, from the triangle ABD,

$$BD = c \sin A, \quad . \quad (29)$$

and, from the triangle BCD,

$$BD = a \sin C. \quad . \quad (30)$$

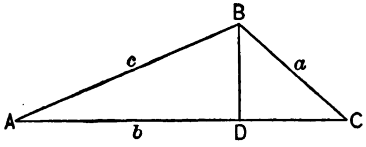


FIG. 10.

$$\therefore \quad CD = AC - AD = b - c \cos A. \quad . \quad . \quad . \quad (31)$$

Equating (29) and (30) gives

$$\left.\begin{aligned}
&c \sin A = a \sin c, \quad \text{or} \quad \frac{a}{c} = \frac{\sin A}{\sin C}.\\
&\text{Similarly, or by analogy,}\\
&\frac{a}{b} = \frac{\sin A}{\sin B}, \quad \text{and} \quad \frac{b}{c} = \frac{\sin B}{\sin C}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (32)$$

From the figure,

$$a^2 = (BD)^2 + (CD)^2.$$

Substituting for BD and CD from (29) and (31), we have

$$a^2 = c^2 \sin^2 A + b^2 - 2bc \cos A + c^2 \cos^2 A;$$

or, since $\sin^2 A + \cos^2 A = 1$,

$$\left.\begin{aligned}
&a^2 = b^2 + c^2 - 2bc \cos A, \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.\\
&\text{Similarly, or by analogy,}\\
&\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \quad \text{and} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.
\end{aligned}\right\} \quad (33)$$

From (29) and (31) we also have

$$\left.\begin{aligned}
&\tan C = \frac{BD}{CD} = \frac{c \sin A}{b - c \cos A} = \frac{\sin A}{\frac{b}{c} - \cos A};\\
&\tan B = \frac{\sin C}{\frac{a}{b} - \cos C};\\
&\tan A = \frac{\sin B}{\frac{c}{a} - \cos B}.
\end{aligned}\right\} \quad . \quad (34)$$

18. Let ABC, Fig. 11, represent a plane triangle, the parts being represented as usual.

Take $CE = CA$, and draw AD and EH perpendicular to AE. We have

$$CAE + CEA = 180° - C = A + B. \quad \therefore \; CAE = CEA = \tfrac{1}{2}(A + B),$$

and $\quad BAE = CEA - CBA = \frac{1}{2}(A + B) - B = \frac{1}{2}(A - B).$

Also $CAD = 90° - CAE$, and $CDA = 90° - (AEC = CAE)$.

Hence $$CD = AC = b.$$

Now

$$\frac{a+b}{a-b} = \frac{BD}{BE} = \frac{AD}{EH} = \frac{AE \tan \frac{1}{2}(A+B)}{AE \tan \frac{1}{2}(A-B)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}. \quad (35)$$

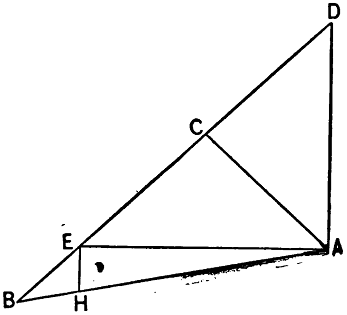


FIG 11.

From triangle ABE,

$$\frac{BE}{AB} = \frac{\sin BAE}{\sin AEB},$$

or

$$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

In triangle ABD,

$$\frac{BD}{AB} = \frac{a+b}{c} = \frac{\sin BAD}{\sin ADB} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}. \quad . \quad . \quad (37)$$

Eq. (37) divided by (36) also gives

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \quad . \quad . \quad . \quad . \quad . \quad (38)$$

which furnishes another demonstration for (35).

A slight variation of the above solution of the tangent problem was given by the author in Vol. I, No. 1, of *The American Mathematical Monthly*. It has since found its way into text-books on igonometry. Still another solution by the author may be seen l. III, No. 11, of the same journal.

SOLUTION OF PLANE OBLIQUE TRIANGLES.

19. There are three cases.

CASE I.—In this case two of the given parts are a side, and the angle opposite; the other part being either a side or an angle.

Example 1.—Let A, a, and B, Fig. 10, be given.

$$C = 180 - (A + B).$$

From (32),

$$b = a\frac{\sin B}{\sin A}; \quad \text{similarly} \quad c = a\frac{\sin C}{\sin A}.$$

Example 2.—A, a, and b given.

(32) gives $\sin B = \sin A\frac{b}{a}$; then the table gives B.

Now $C = 180° - (A + B)$; and $c = a\frac{\sin C}{\sin A}$.

CASE II.—Given two sides and the included angle.

Let b, c, and A be given. We have, from (34),

$$\tan C = \frac{\sin A}{\frac{b}{c} - \cos A}.$$

Then

$$B = 180° - (A + C); \quad \text{and} \quad a = b\frac{\sin A}{\sin B}.$$

Or, from (33),

$$a = (b^2 + c^2 - 2bc \cos A)^{\frac{1}{2}}.$$

Then

$$\sin B = \sin A\frac{b}{a}, \quad \text{and} \quad C = 180° - (A + B).$$

CASE III.—Given the three sides a, b, and c.

Eq. (33) gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Then

$$\sin B = \sin A\frac{b}{a}, \quad \text{and} \quad C = 180° - (A + B).$$

The above formulas are all-sufficient for all practical purposes. This chapter constitutes a complete treatise on trigonometry, though the deductions from it are endless, as the examples in arithmetic are endless.

TABLE FOR SOLUTION OF OBLIQUE TRIANGLES.

(See Fig. 10.)

Given.	Required.	Formulas.
6. A, B, a	C, b, c	$b = \frac{a \sin B}{\sin A}$, $\quad c = \frac{a \sin (A + B)}{\sin A}$.
7. A, a, b	B, C, c	$\sin B = \frac{b \sin A}{a}$, $\quad c = \frac{a \sin (A + B)}{\sin A}$.
8. A, b, c	B, C, a	$\tan B = \frac{\sin A}{\frac{c}{b} - \cos A}$, $\quad a = \frac{b \sin A}{\sin B}$.
9. a, b, c	A, B, C	$\text{vers} A = \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)(a+c-b)}{2bc}$, $\sin B = \frac{b \sin A}{a}$.

Use the first form for vers A with a table of squares, the second without; they are the best formulas known for this case.

The following are the best formulas known for the area.

10. Area $\frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$.

It is never necessary to compute but *one* unknown part, and in the 3d case none at all, to have the required data for one of these equations; and the computation is shorter than by any other formula.

Observe that $a \sin C$ is equal to the perpendicular from B upon b, $b \sin C$ is equal to the perpendicular from A upon a, etc.

Trigonometric Formulas.

11. $\sin A = \sqrt{1 - \cos^2 A} = \dfrac{1}{\operatorname{cosec} A} = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A.$

12. $\cos A = \sqrt{1 - \sin^2 A} = \dfrac{1}{\sec A} = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A$

$= 2 \cos^2 \tfrac{1}{2}A - 1 = 1 - 2 \sin^2 \tfrac{1}{2}A.$

13. $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{1}{\cot A} = \operatorname{cosec} 2A - \cot 2A$

$= \dfrac{1 - \cos 2A}{\sin 2A} = \dfrac{\sin 2A}{1 + \cos 2A}.$

14. $\cot A = \operatorname{cosec} 2A + \cot 2A =$ the reciprocal of any expression for $\tan A$.

15. $\sec A = \dfrac{1}{\cos A} =$ the reciprocal of any expression for $\cos A$.

16. $\operatorname{cosec} A = \dfrac{1}{\sin A} =$ the reciprocal of any expression for $\sin A$.

17. $\operatorname{vers} A = 1 - \cos A = 2 \sin^2 \tfrac{1}{2}A.$

18. $\operatorname{exsec} A = \sec A - 1 = \dfrac{\operatorname{vers} A}{\cos A}.$

19. $\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B.$

20. $\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B.$

21. $\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$

22. $\sin A - \sin B = 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).$

23. $\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B).$

24. $\cos B - \cos A = 2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B).$

25. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin (A + B) \sin (A - B).$

26. $\cos^2 A - \sin^2 B = \cos (A + B) \cos (A - B).$

27. $\tan A \pm \tan B = \dfrac{\sin (A \pm B)}{\cos A \cos B}.$

28. $\cot A \pm \cot B = \dfrac{\pm \sin (A \pm B)}{\sin A \sin B}.$

The above formulas contain the practical general relations existing among the functions of an angle. By writing $\frac{1}{2}A$, $2A$, etc., in place of A, by repetitions, etc., the formulas may be greatly multiplied without producing any new relations. This practice is too common in Trigonometries, Field-books, etc.

CHAPTER IV.

SIMPLE CURVES CONNECTING RIGHT LINES.

LET $ABCDE$ represent a circular arc joining the straight lines AV and EV, which are tangent to the curve at A and E.

AV and EV are tangents to the curve, A and E are tangent points, and the angle KVE is the angle of intersection, and shows the change of direction in passing from one tangent to the other.

V is the point of intersection, or vertex.

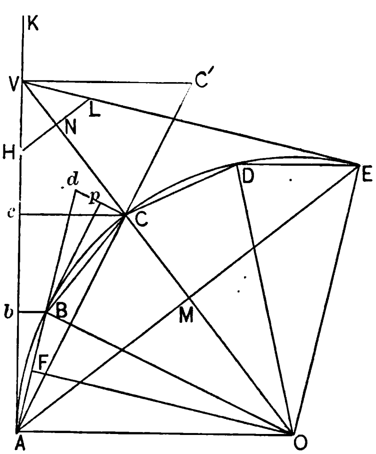


FIG. 12.

PROPERTIES RELATING TO THE CIRCLE.

The following propositions rest upon elementary geometrical principles, may be regarded, for the most part, as axiomatic.

(*a*) A tangent to a circle is perpendicular to the radius, at the point of contact.

(*b*) Tangents drawn to the circle from the same point are equal, and the angle between these tangents, and the chord joining the tangent points, are equal. Thus,

$$AV = EV, \text{ and } VAE = VEA.$$

(*c*) The central angle AOE, subtended by a chord, is equal to the exterior angle KVE between two tangents to the curve at the extremities of the chord.

(*d*) The angle between a tangent and chord is equal to the angle subtended at the circumference by the same or by an equal chord. Thus, $VAB = BCA = BAC$, etc.

(*e*) An angle between a tangent and chord, or an angle subtended at the circumference by that chord, is equal to one half the central angle subtended by the same chord. Thus,

$$VAB = BCA = \tfrac{1}{2}AOB.$$

(*f*) Equal chords subtend equal angles at the center of a circle, and also at the circumference. Thus, if $AB = BC$, etc.,

$$ACB = BAC, \text{ etc.,} \quad \text{and} \quad AOB = BOC, \text{ etc.}$$

(*g*) A radius perpendicular to a chord bisects the chord, and also the angle and the arc subtended by the chord. Thus, if OC is perpendicular to AE, we have

$$AM = ME, \quad AOV = EOV,$$

and

$$AC = CE.$$

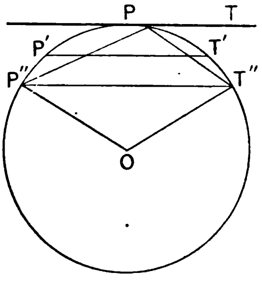


FIG. 13.

(*h*) Parallel chords, or a tangent and parallel chord, intercept equal arcs. Thus, in Fig. 13, if PT, $P'T'$, and $P''T''$ are parallel, then $P'P''$ and $T'T''$ are equal, also PP' and TT' are equal.

(*i*) The exterior or deflection angle KBC, Fig. 14, between any two chords AB and BC is equal to half the central angle AOC, subtended by the chords. This is easily shown as follows:

1. Join AC. $BAC = \frac{1}{2}BOC$, and $BCA = \frac{1}{2}AOB$, as stated above. But

$$KBC = BAC + BCA;$$

$$\therefore \; KBC = \tfrac{1}{2}AOB + \tfrac{1}{2}BOC$$

$$= \tfrac{1}{2}AOC.$$

2. Draw the tangent HBL. Then

$$KBC = KBL + LBC$$

$$= HBA + LBC$$

$$= \tfrac{1}{2}AOB + \tfrac{1}{2}BOC = \tfrac{1}{2}AOC.$$

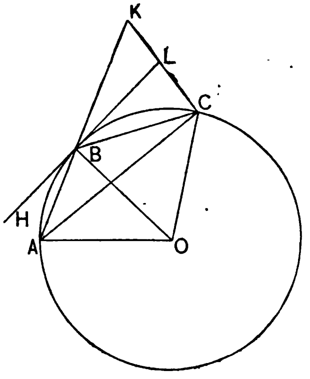


FIG. 14.

3. Since the chords AB and BC are parallel to tangents drawn at the *middle* of the arcs AB and BC, it is evident that, in passing from one chord to the other, we turn through an angle measured by one half the arc AB + one half the arc BC, that is, through an angle equal to one half AOC.

SOME ELEMENTARY RELATIONS.

In Fig. 12 drop the perpendiculars Bb, Cc, etc., upon the tangent AV.

Ab, bc, etc., are called tangent distances, and Bb, Cc, etc., tangent offsets.

Prolong AB, making $Bd = AB$. dC is called the chord offset.

Then, from the preceding article, we have

$$CBd = \tfrac{1}{2}AOC = AOB.$$

Drop the perpendicular Bp upon Cd. This bisects the angle CBd and the base Cd. Hence

$$CBp = dBp = \tfrac{1}{2}CBd = \tfrac{1}{2}AOB = BAb.$$

Since, therefore, the triangles CBp, dBp, and BAb have an acute angle in each equal, and the hypothenuses also equal, they are equal in all respects; and therefore

$$Cp = pd = Bb; \quad \text{also} \quad Cd = 2Bb.$$

Since Ab is tangent to the curve at A, Bp is likewise tangent to the curve at B.

Represent the radius AO by R, the tangent, or vertex distance, AV by T, and the angle of intersection KVE, as well as the central angle AOE, by V. Represent the chords AB, BC, etc., by c, a long chord, as AC, by C, the tangent distances Ab, bc, etc., by d, d_1, etc., and the tangent offsets bB, cC, etc., by t, t_1, etc. Then the chord offset $Cd = 2t$. Represent the vertex distance CV by E, and the middle ordinate of a long chord by M. Let L represent the length of the curve, $P.C.$ (Point of Curve) the beginning of the curve, and $P.T.$ (Point of Tangent) the end of the curve.

The degree of a curve has been defined as the number of degrees subtended at the center by a chord 100 feet in length. This definition of curvature is, however, awkward, arbitrary, and false. It is founded on error; it involves unnecessary labor and ends in anomalous and erroneous results.

The degree of a curve may be defined as the change in its direction between one point, and another 100 feet from the first, measured on the curve. This is the change in direction which one would make in moving 100 feet on the curve from one point to another. Or, as the angle subtended at the center of the curve by an arc 100 feet in length.

Let D = the degree of the curve.

The circumference of a 1° curve is therefore $360 \times 100 = 36000$ feet, and its radius is $\frac{36000}{2\pi} = 5729.578$ feet, almost exactly.

Since a 2° curve changes its direction 2° in 100 feet, its circumference is only half that of a 1° curve; and since the radius varies directly with the circumference, the radius, too, is only half that of a 1° curve. For the same reason the radius of a 3° curve is precisely $\frac{1}{3}$ the radius of a 1° curve; and, generally, the radius of a D° curve is exactly equal to the radius of a 1° curve divided by D; it is therefore equal to $\frac{5729.578}{D}$.

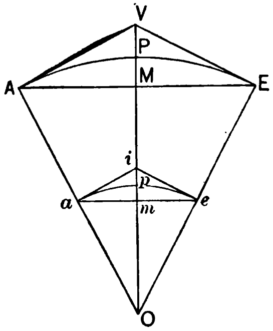


FIG. 15.

Referring to Fig. 15, we see that for the same central angle AOE, the arc, the tangent, the chord, the middle ordinate, the external secant, etc., vary directly with the radius or inversely with the degree of curvature. For example: If $aO = \frac{1}{2}AO$, then $ape = \frac{1}{2}APE$, $ai = \frac{1}{2}AV$, etc. If APE is a 1° or 2° or 3° curve of length L, then ape is a 2° or 4° or 6° curve of length $\frac{1}{2}L$.

Hence to compute any function of any curve, the tangent or external, for example, from the corresponding function of another curve, it is only necessary to multiply or divide, as the case may be, by the ratio of their radii or degrees of curvature.

Example.—Find the tangent of a 7° 13′ = 433′ curve, the cen- ngle being 37° 50′.

The tangent of a 1′ curve, by Table VII, is 117812. Then

$$T = 117812 \div 433 = 272.1.$$

Using a table giving functions of a 1° curve, we have:

Tangent of a 1° curve = 1963.6.

Then Tangent of a 1′ curve = 1963.6 × 60 = 117816.

Finally, $117816 \div 433 = 272.1.$

Or, $13' \div 60 = 0^\circ.216,$

and therefore $7^\circ\ 13' = 7^\circ.216.$

Then $1963.6 \div 7.216 = 272.1.$

With a table giving functions of a 1° curve there is no escape from dividing by 60, which division is obviated by giving the functions of a 1′ curve as in Table VII.

It is often desirable to know the difference in the lengths of a chord and its subtended arc, and for this purpose we deduce the following formula:

$$d = .001269239249\frac{D^2}{n^3} - .0000000048329\frac{D^4}{n^5}. \quad . \quad . \quad (8)$$

$$= .001269\frac{D^2}{n^3} - .0000000048\frac{D^4}{n^5}, \text{ very nearly,} \quad . \quad (9)$$

$$= .001269\frac{D^2}{n^3}, \text{ approximately.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In these equations D = the degree of curvature, d = the difference between any chord and the subtended arc, and n = arc of 100 feet divided by this arc.

For an arc of 100 feet $n = 1$ and $d = .001269D^2$, nearly; . (11)
For an arc of 50 feet $n = 2$ and $d = .000158D^2$, nearly; . (12)
For an arc of 25 feet $n = 4$ and $d = .00002D^2$, nearly. . (13)

For n sub-chords per station the sum of the difference per station is

$$nd = .001269\frac{D^2}{n^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Representing the central angle of the curve by V, and the number of stations in the curve by N, we have $N = \frac{V}{D}$; and hence the length of the curve exceeds the sum of the lengths of the sub-chords by

$$E = Nnd = .001269\frac{D^2}{n^2} \times \frac{V}{D} = .001269\frac{VD}{n^2}. \quad . \quad (14')$$

Hence, in laying out curves, $\frac{D}{n^2}$ should be nearly constant.

Since in a 4° curve the chord of an arc of 100 feet is 99.98 feet, curves from 0° to 4° can be properly laid out with chords of 100 feet, and with the same degree of accuracy we may lay out curves from 4° to 16° with chords of 50 feet, and those from 16° to 64° with chords of 25 feet.

Very sharp curves can be easily laid out by swinging a chain around, while one end is held at the center of the curve.

Let s represent any arc, c its chord, and c_1 the chord of one half of the arc s.

Then, from above,

$$s - c = 8\left(\frac{s}{2} - c_1\right) = 4s - 8c_1, \quad \text{or} \quad s = \frac{8c_1 - c}{3}.$$

This is said to be Huygens' approximation to the length of an arc. . . . .

The following table shows the differences between arcs of 25 feet and of 50 feet, and the chords of those arcs. (See Fig. 16.)

Deg. D	Arc 25	Arc 50	Deg. D	Arc 25	Arc 50	Deg. D	Arc 25	Arc 50
1	.000	.000	11	.002	.019	21	.009	.070
2	.000	.001	12	.003	.023	22	.010	.077
3	.000	.001	13	.003	.027	23	.010	.084
4	.000	.003	14	.004	.031	24	.011	.091
5	.000	.004	15	.004	.036	25	.012	.099
6	.001	.006	16	.005	.041	26	.013	.107
7	.001	.008	17	.006	.046	27	.014	.116
8	.001	.010	18	.006	.052	28	.016	.124
9	.002	.013	19	.007	.058	29	.017	.133
10	.002	.016	20	.008	.064	30	.018	.143

II. Suppose the *chord* $AB = 100$ feet, and $AOB = D_1$ (Fig. 16). Then

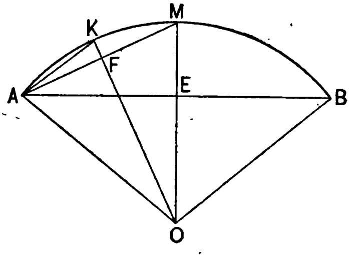


FIG. 16.

$$AOM = \frac{D_1}{2},\quad MAE = \frac{D_1}{4}, \text{ etc.}$$

Hence

$$AM = AE \sec MAE$$

$$= 50 \sec \frac{D_1}{4}.$$

Therefore

$$AM - 50 = 50\left(\sec \frac{D_1}{4} - 1\right)$$

$$= 50 \operatorname{exsec} \frac{D_1}{4}.$$

Similarly, $AK = AF \sec \frac{D_1}{8} = 25 \sec \frac{D_1}{4} \sec \frac{D_1}{8}$, etc., etc.

The following table gives the excess of AK over 25 feet, and of AM over 50 feet, when chord AB is 100 feet.

Deg. D_1	$AK - 25$	$AM - 50$	Deg. D_1	$AK - 25$	$AM - 50$	Deg. D_1	$AK - 25$	$AM - 50$
1	.000	.000	11	.036	.058	21	.131	.211
2	.001	.002	12	.043	.069	22	.144	.231
3	.003	.004	13	.050	.081	23	.157	.252
4	.005	.008	14	.058	.093	24	.171	.275
5	.007	.012	15	.067	.101	25	.186	.299
6	.011	.017	16	.076	.122	26	.201	.323
7	.015	.023	17	.086	.138	27	.217	.349
8	.019	.030	18	.096	.155	28	.233	.375
9	.024	.039	19	.107	.172	29	.250	.403
10	.030	.048	20	.119	.181	30	.268	.431

Comparing the preceding tables we learn that, when the chord AB is 100 feet, the chord AM differs nearly four times as much from 50 feet as the chord of the *arc* of 50 feet differs from 50 feet. Furthermore, that the chord AK differs nearly sixteen times as much from 25 feet as the chord of the *arc* of 25 feet differs from 25 feet.

Thus let us first suppose AMB (Fig. 16) to be a 10° curve, and the chord $AB = 100$ feet. Then by the table $AK = 25.030$ feet, and this would lead to an error of $.030 \times 40 = 1.2$ feet in laying out a curve $25 \times 40 = 1000$ feet long, taking chord $AK = 25$ feet long.

Suppose, secondly, that the arc $AMB = 100$ feet and therefore arc $AK = 25$ feet, and the chord AK is but .002 of a foot less than 25 feet.

Hence in laying out the above curve, taking the chord $AK = 25$ feet, the error committed would be only $.002 \times 40 = .08$ of a foot.

Thus we see that when the chord of a station is taken = 100 feet, the sub-chords AM, AK, etc., differ so much from 50 feet, 25 feet, etc., as to largely vitiate the results, whereas such is by no means the case when the arc AMB is taken = 100 feet.

Indeed, when the chord AB is 100 feet, the *shorter* the sub-chords used in laying out a curve, the *greater* the discrepancy in the measurement, on this basis. Thus for a 10° curve the difference between *four* equal sub-chords and 100 feet is $.030 \times 4 = .120$ of a foot; whereas the difference between *two* equal sub-chords and 100 feet is $.048 \times 2 = .096$ of a foot. The reverse is of course the case when the *arc* AMB is made the standard of measurement.

These facts are evident; for when the chord is made the standard of measurement, the sum of the lengths of the sub-chords *exceeds more and more* the length of the chord, the shorter they are; whereas when the arc is the standard of measurement, the sum of the lengths of the sub-chords *falls short* of the length of the arc *less and less* the shorter they are.

Most recent writers have endeavored to obviate the inconveniences and inconsistencies above pointed out by inconsistent assumptions, such as basing the curves of different degrees upon chords of different lengths. This scheme gives the values of some of the radii quite correct; but it causes sudden breaks in the values where the changes are made.

For example, how can there be two different values (819.02 and 818.64) of the radius of a 7° curve?

And why should the radius of a 7° 10′ curve be given quite correct, while that of a 6° 50′ curve is quite incorrect?

Again, we are told by a recent author that, in *practice*, it is *customary* to take the radius of a 1° curve as 5730 feet, and to assume the radius to vary inversely as the degree. Thus for a 4° curve the radius would be $\frac{5730}{4} = 1432.5$ feet. This is rational and, moreover, is precisely what is here advocated, except that the true value of the radius of a 1° curve is used, viz., 5729.58 feet.

FORMULAS.

From the triangle AOV, Fig. 12, we have

$$AV = AO \tan AOV, \quad \text{or} \quad T = R \tan \tfrac{1}{2}V. \quad . \quad . \quad (15)$$

Transposing, we have

$$R = \frac{T}{\tan \frac{1}{2}V} = T \cot \tfrac{1}{2}V. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

From the triangle AOF, we find

$$AO = \frac{AF}{\sin AOF}, \quad \text{or} \quad R = \frac{\frac{1}{2}c}{\sin \frac{1}{2}D} = \tfrac{1}{2}c \operatorname{cosec} \tfrac{1}{2}D. \quad . \quad (17)$$

This value of R in (15) gives

$$T = \frac{\frac{1}{2}c \tan \frac{1}{2}V}{\sin \frac{1}{2}D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Measure equal distances VH and VL along the tangents in Fig. 12 and draw HNL.

Measure HN and VN.

Now,
$$\frac{AV}{AO} = \frac{VN}{HN}, \quad \text{or} \quad AV = AO\frac{VN}{HN};$$

that is,
$$T = R\frac{VN}{HN}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If AM and VM are measured, then

$$T = R\frac{VM}{AM}, \quad \text{or} \quad R = T\frac{AM}{VM} \quad . \quad . \quad . \quad . \quad (20)$$

Eqs. (19) and (20) serve to fix geometrically, without measuring angles, the tangent point of a curve of given radius that will unite two straight lines on the ground.

In Fig. 12 draw VC' perpendicular to AV to meet AC prolonged. Now the angle $AVC = \frac{1}{2}AVE = \frac{1}{2}(180 - V) = 90 - \frac{1}{2}V$. Hence

$$VCC' = ACO = 90^\circ - BOC = 90^\circ - \tfrac{1}{2}V.$$

and [illegible] $= 90° - \frac{1}{2}I = 90° - \frac{1}{2}I$

[illegible] and [illegible] $= E$

Hence

$$\frac{E}{T} = [illegible] = \tan \tfrac{1}{4}I, \quad \text{or} \quad E = T \tan \tfrac{1}{4}I. \quad . \quad . \quad (21)$$

Substituting $T = R \tan \frac{1}{2}I$ [illegible] gives

$$E = R \tan \tfrac{1}{2}I \tan \tfrac{1}{4}I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21')$$

Since the triangles [illegible] and [illegible] are similar, we have

$$\frac{Cd}{BC} = \frac{BC}{BO}, \quad \text{or} \quad Cd = \frac{BC^2}{BO};$$

or

$$Cd = 2t = \frac{c^2}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$Bb = t = \frac{c^2}{2R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

We also have

$$Bb = AB \sin BAb, \quad \text{or} \quad t = c \sin \tfrac{1}{2}D, \quad . \quad . \quad . \quad (24)$$

and

$$2t = 2c \sin \tfrac{1}{2}D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Example.—Given $R = 1909.9$ feet and $c = 100$ feet, to find the tangent and the chord offsets for 100 feet.

By (23),

$$t = \frac{10000}{3819.8} = 2.618, \quad \text{and} \quad 2t = 5.236,$$

or Table I shows that 1909.9 is the radius of a 3° curve, and

$$\therefore \quad \tfrac{1}{2}D = 1° \; 30', \quad \text{and} \quad \sin \tfrac{1}{2}D = .02618.$$

Then, by (24),

$$t = .02618 \times 100 = 2.618, \quad \text{and} \quad 2t = 5.236.$$

The tangent offsets are given in Table I, and the chord offsets are twice the tangent offsets.

In laying out curves, the chain is stretched from point to point on the curve, and coincides, therefore, with chords of the curve.

Since the process is the same whatever the length of chain used, we will assume it to be 100 feet long.

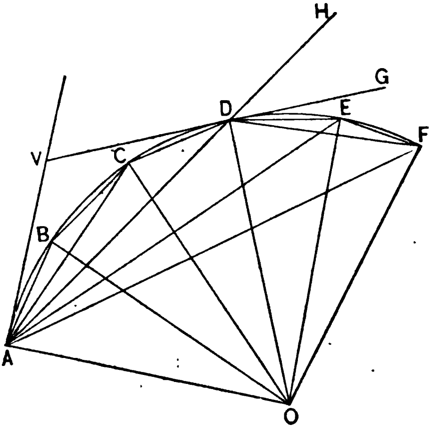


FIG. 17.

The length of the curve is expressed in chains, in terms of the central angle $AOF = V$, and the degree of the curve $AOB = D$. The number of chains is evidently equal to $\frac{V}{D}$.

Thus, in the figure, if $V = 23°$ and $D = 5°$, the curve is $\frac{23}{5} = 4\frac{3}{5}$ chains, or 460 feet long. As the angle $AOE = 5° \times 4 = 20°$, $EOF = 23° - 20° = 3°$; and the arc EF is $\frac{3}{5} \times 100 = 60$ feet long.

It is usual, in laying out curves, to assume the radius R, and to find the degree of the curve D from it; or to assume D (usually in degrees and minutes), and to find R from it. Neither way is best.

To assume a value of R or of D does not aid in the least in properly locating the curve. Generally the surface of the ground does indicate approximately the position of the curve, and the proper course to pursue, therefore, is the following: Divide the tangent of a one-minute curve by the length in round numbers of the desired tangent, and neglect the decimal in the quotient. This gives the degree of the required curve in minutes. We may change the quotient to an even number of minutes, or to some multiple of 10 if we wish, if there is sufficient latitude to be taken in the position of the P.C.

Then divide the tangent of a one-minute curve by the corrected quotient for the tangent required.

Example.—$V = 46° 30'$, and the tangent should be 1100 feet or over. Find the degree of the curve, and the length of the tangent.

Dividing the tangent of a 1′ curve by 1100 gives

$$147697 \div 1100 = 134' = 2° 14'.$$

Now $2° 10' = 130'$, and $147697 \div 130 = 1136.13$, the tangent required.

LONG CHORDS AND ORDINATES TO LONG CHORDS.

Let A, B, C, etc., represent stations upon a curve.

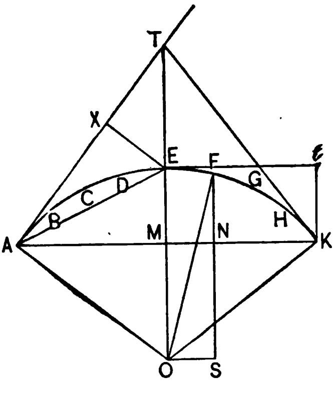


FIG. 18.

Draw the lines as represented, EX being a perpendicular from the middle of the curve E upon the tangent AT. Let $M = EM$, $M' = FN$, etc.

Draw AE, and we have the angle EAM equal to the angle EAX, and therefore the triangles EAM and EAX are equal in all respects. From this we learn that the tangent offset EX for any arc AE is equal to the versin EM of that arc, or to the middle ordinate EM of the chord AK of twice the arc.

Or, draw the tangent Et and the perpendicular Kt upon it. Then the tangent offset Kt of the arc EK = the versin EM of the same arc = the middle ordinate EM of the arc AEK.

To find the middle ordinate, M. (See Fig. 18.)

1°. $M = EM = EX = AM \text{ tangent } EAM = \frac{1}{2}C \tan \frac{1}{4}V$. . (26)

2°. $M = EM = EX = ET \cos TEX = E \cos \frac{1}{4}V$. . . . (26′)

3°. $M = EO - MO = R - R\cos\frac{1}{2}V = R(1-\cos\frac{1}{2}V)$

$$= R \text{ vers } \tfrac{1}{2}V. \quad . \quad . \quad (26'')$$

4°. $M = EO - MO = EO - \sqrt{AO^2 - AM^2} = R - \sqrt{R^2 - \frac{1}{4}C^2}$. . (26‴)

To find any ordinate FN distant d from the center of the chord. Prolong FN, to meet OS, drawn parallel to AK, and join FO.

Now, $FS = \sqrt{FO^2 - OS^2} = \sqrt{R^2 - d^2}$. $NS = MO = R - M$.

$$\therefore\ FN = \sqrt{R^2 - d^2} - (R - M) = M + \sqrt{R^2 - d^2} - R.$$

Other methods will be given in connection with laying out curves by ordinates from a long chord.

Approximate Values of Ordinates to Short Chords.

Divide the chord into any number of equal parts, eight for example, at i, k, l, etc., and erect the ordinates Em, Fn, etc., and prolong them to meet the curve in E', F', etc. Let $Em = m$, $Fn = m'$, etc. We have, from geometry,

$$Em = \frac{Am \times mB}{mE'} = \frac{\frac{1}{2}c \times \frac{1}{2}c}{2R - m};$$

or, approximately,

$$m = \frac{\frac{1}{2}c \times \frac{1}{2}c}{2R} = \frac{c^2}{8R}. \quad . \quad . \quad (27)$$

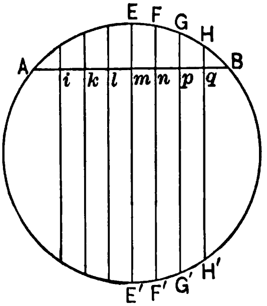


FIG. 19.

Similarly,

$$Fn = m' = \frac{\frac{5}{8}c \times \frac{3}{8}c}{2R} = \frac{15}{16} \cdot \frac{c^2}{8R} = \tfrac{15}{16}m = m - \tfrac{1}{16}m; \quad . \quad . \quad (28)$$

$$Gp = m'' = \frac{\frac{3}{4}c \times \frac{1}{4}c}{2R} = \frac{12}{16} \cdot \frac{c^2}{8R} = \tfrac{12}{16}m = m - \tfrac{4}{16}m; \quad . \quad . \quad (28')$$

$$Hq = m''' = \frac{\frac{7}{8}c \times \frac{1}{8}c}{2R} = \frac{7}{16} \cdot \frac{c^2}{8R} = \tfrac{7}{16}m = m - \tfrac{9}{16}m. \quad . \quad . \quad (28'')$$

For any other equal divisions of the chord we have similar results.

To establish the points E, F, etc.

Set E equally distant from A and B, and at the *distance* m from the point m; then F, equally distant from k and B, and $\frac{15}{16}m$ from n; then G, equally distant from m and B, and $\frac{3}{4}m$ from p; lastly, H, equally distant from p and B, and $\frac{7}{16}m$ from q.

This method involves much less labor than that of drawing sub-chords to find the points F, G, etc. If we draw a tangent at E, the offsets to F, G, H, etc., will be, according to the preceding formulas, $\frac{1}{16}m$, $\frac{4}{16}m$, $\frac{9}{16}m$, etc. This shows a convenient way of finding the points on the curve.

To compute tangent offsets and middle ordinates by means of a series.

In a way similar to that pursued in finding the difference between an arc and its chord we find

$$t = .872664625997D - .00002215240389D^3 + .000000000224933360386D^5 -, . \qquad (29)$$

or

$$t = .872664626D - .000022152404D^3 + .000000000224933604D^5. \qquad (30)$$

We may find the tangent offset for m stations by multiplying the successive terms of (30) by m^2, m^4, etc.

We thus find tang offset for arc of 50 feet, or $\frac{m}{2}$ stations,

$$t' = .218166156499D - .0000013845252 4D^3 + .00000000000351458756D^5 -. \; . \qquad (31)$$

For 1 foot $m^2 = (.01)^2 = .0001$, $m^4 = .00000001$, etc. Then,

The tang offset for 1 ft.

$$= t_1 = .0000872664626D - .0000000000000221524 04D^3 + . \qquad (32)$$

From (30), we have $t = .87D$, approximately.

$$\text{For } n \text{ stations } tn = .87n^2D. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (33)$$

Wellington in *Railway Location*, ch. xxx, recommends the equation

$$t = \tfrac{7}{8}n^2D, \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (33')$$

which is practically the same as (33). (33) is a trifle more accurate than (33′), but either is sufficiently accurate for all cases in which n^2D does not much exceed thirty.

The same formula gives the middle ordinate, n representing the number of stations on either side of the center, or $2n$ the number of stations in the arc. Any other ordinates desired are then given by eqs. (27), (28), etc.

Example.—Find six offsets of a 3° curve at points 50 feet apart. (See Fig. 18)

$$\begin{aligned}
\text{For } n &= \tfrac{1}{2}, & t &= \tfrac{7}{8} \times \tfrac{1}{4} \times 3 = .66;\\
n &= 1, & t &= \tfrac{7}{8} \times 1 \times 3 = 2.62;\\
n &= \tfrac{3}{2}, & t &= \tfrac{7}{8} \times \tfrac{9}{4} \times 3 = 5.91;\\
n &= 2, & t &= \tfrac{7}{8} \times 4 \times 3 = 10.50;\\
n &= 2\tfrac{1}{2}, & t &= \tfrac{7}{8} \times \tfrac{25}{4} \times 3 = 16.41;\\
n &= 3, & t &= \tfrac{7}{8} \times 9 \times 3 = 23.62.
\end{aligned}$$

These results are of course equal to the middle ordinates for 1, 2, 3 . . . 6 stations of the same curve.

Laying out Curves.

Since in laying out curves the operation or method is the same whatever the length of the chain or chord, we will here assume it to be 100 feet long.

A. *By Deflection Angles.*

Let A in Fig. 17 be the P.C. Set the instrument at A, and turn off from the tangent AV the given deflection angle $VAB = \frac{1}{2}D$, D being the degree of curve. This will give the direction, AB, and measuring 100 feet in this direction, the point B will be determined. Turn off the additional angle $BAC = \frac{1}{2}D$, the telescope being now directed toward C, and set C in the line AC and 100 feet from B. Turn off the additional angle $CAD = \frac{1}{2}D$, and set D in the line AD and 100 feet from C. Proceed in the same way for other stations to the end of the curve, or so far as the stations can be seen from A.

It is usually impossible, on account of obstructions likely to be met with, to lay out the whole of a curve from the first station. When such is the case, we determine as many stations as convenient, remove the instrument to the last station so determined, and proceed from that as from the first station. For example : Suppose B, C, and D to be found with the instrument at A. Remove the instrument to D, sight to A, turn off the angle $ADV = DAV = \frac{3}{2}D$, and the line of sight will be in the direction of the new tangent DV at D. Reverse the telescope, and the line of sight will point forward along the same tangent VDG. Now set E, F, etc., from the tangent DG, as B, C, and D were set from the tangent AV.

In setting the first stake from the new position of the instrument, as E from D, no notice need be taken of the tangent at D, it being necessary simply to turn off from the line ADH the angle $HDE = HDG + GDE = ADV + GDE = 4(\frac{1}{2}D)$.

In the new position of the instrument we observe that, in all cases, the deflection *from* the line pointing to the back station *to* the line pointing to the forward station is as many times the deflection angle $\frac{1}{2}D$ as there are chains in the curve between the back and the forward station. Of course this applies to simple curves only.

The beginning of a curve, as well as the end, usually falls between regular stations, giving short chords at the ends. The deflection angle for a short arc is such a part of the full deflection angle as the short arc is of the full arc.

Let a represent the arc AB or BC, etc., and a_1 represent the arc EF. Also $EOF = D_1$.

Hence $DFE = \frac{1}{2}D$, and $EDF = \frac{1}{2}D_1$, and therefore

$$\frac{\frac{1}{2}D_1}{\frac{1}{2}D} = \frac{a_1}{a}, \quad \text{or} \quad \frac{1}{2}D_1 = \frac{1}{2}D\frac{a_1}{a}, \quad \text{or} \quad a_1 = \frac{\frac{1}{2}D_1}{\frac{1}{2}D}a.$$

In beginning a curve, the instrument being at the first station, it is convenient to place the zeros of the instrument plates together, and direct the line of collimation along the tangent to the curve. The reading on the limb for any station will then be equal to the total deflection angle for that station.

If the vernier is not disturbed while laying out the curve, it is plain that when the instrument is moved to its second position D, Fig. 17, and the line of sight directed to A, the reading will be equal to the total deflection from A to D; and that, an additional angle equal to this deflection being turned off, the reading will be equal to the central angle AOD, and the line of sight will be in the direction of the tangent at D. The same is true for all positions of the instrument. Since the deflections for the last section of the curve, that is, from the last position of the instrument to the end of the curve, is turned off but once, the reading on the instrument, when the curve is finished, will be equal to the total central angle V, less the deflection for the last section. This furnishes a convenient check for the work.

B. *By Tangent Offsets. New Method.*

Let $ABCDEFGH$ represent a curve having a short chord $AB = c'$ subtending an angle $AOB = D_1$ at one end of the curve.

Define the tangent AV by setting stakes upon it. Draw Bc, Cd, etc., parallel to AV, and BX, CcX', etc., perpendicular to AV, or suppose such lines to be drawn. Now

$$BX = AB \sin BAX = c' \sin \tfrac{1}{2} D_1 = t_1,$$

and is given by Table II.

Since the chords BC, CD, etc., are parallel to the tangents at the *middle* of the arcs BC, CD, etc., we have

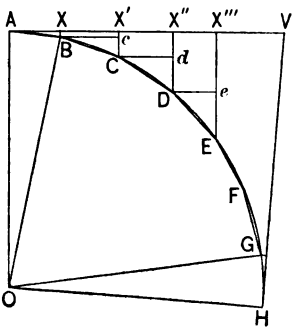


FIG. 20.

$$CBc = D_1 + \tfrac{1}{2}D, \quad DCd = D_1 + \tfrac{3}{2}D, \quad EDe = D_1 + \tfrac{5}{2}D, \text{ etc.}$$

Hence

$$Cc = c \sin (D_1 + \tfrac{1}{2}D), \quad Dd = c \sin (D_1 + \tfrac{3}{2}D),$$
$$Ee = c \sin (D_1 + \tfrac{5}{2}D), \quad \text{etc.}$$

We observe that Cc, Dd, Ee, etc., are respectively the tangent offsets for curves of the degrees $(D_1 + \tfrac{1}{2}D)$, $(D_1 + \tfrac{3}{2}D)$, $(D_1 + \tfrac{5}{2}D)$, etc., and may be taken from Table III. Then

$$CX' = BX + Cc, \ DX'' = CX' + Dd, \ EX''' = DX'' + Ee, \text{ etc.}$$

Set B at a distance c' from A, and t_1 from AV; then C a distance c from B, and CX' from AV; D a distance c from C, and Dx'' from AV, etc.

It will be observed that in this method we avoid constructing a tangent at B, in consequence of the short chord AB; we avoid, secondly, the finding of AX, XX', etc.; thirdly, the errection of perpendiculars, at X, X', etc.; and, lastly, we avoid the use of the radii which are large and fractional. Since c is generally 100, though sometimes an aliquot part of 100, no computation is generally required, and none to mention in any case. Thus we see how simple and short this method is, compared with the method of tangent offsets in general use.

When the curve begins at a station there is no short chord. Then

$$AB = c,\ D_1 = D;\ BX = c \sin \tfrac{1}{2}D,\ Cc = c \sin \tfrac{3}{2}D,\ Dd = c \sin \tfrac{5}{2}D, \text{ etc.}$$

It is best to lay out the curve from each end, so as to avoid offsets inconveniently long. Long offsets may be avoided by drawing a tangent at any station and continuing the curve from that station precisely as from A, when there is no short chord.

To draw a tangent at any station, draw a line through that station and at a perpendicular distance from an adjacent station equal to the tangent offset for a station $= c \sin \frac{1}{2}D$. *Or*, draw it parallel to the chord joining adjacent stations.

It will be noticed that stations on the curve are not opposite stations originally set on the tangent.

The length of the curve gives the number of the station at H.

This is perhaps the best of all methods with out a transit; but a combination of methods is sometimes advisable, as will be shown further on.

We also have

$$AX = C \cos \tfrac{1}{2}D_1,\ XX' = Bc = c \cos (D_1 + \tfrac{1}{2}D), \text{ etc.},$$

or XX', $X'X''$, etc., are respectively equal to the tangent distances for one station of curves of the degrees $D_1 + \frac{1}{2}D$, $D_1 + \frac{3}{2}D$, etc.

These quantities are not needed, however, and it is to be observed that the points X, X', etc., are not established or used.

C. *To Locate a Curve by Ordinates from a Long Chord.*

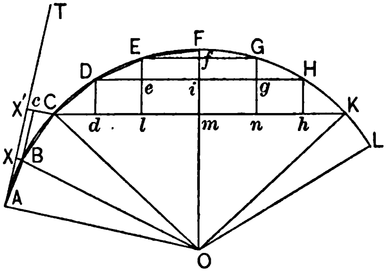


FIG. 21.

Let Fig. 21 represent a curve having an odd number of full chords, and a short chord $AB = c'$ subtending an angle $AOB = D_1$, and a short chord $KL = c''$ subtending an angle $KOL = D_2$. Draw the tangent AT, and drop the perpendiculars BX and CX' upon the tangent. Draw Bc parallel to AT.

Since Bc is perpendicular to AO, and BC is perpendicular to a radius bisecting the angle BOC, the angle $CBc = D_1 + \frac{1}{2}D$.

Now BX is found in Table II; and Cc may be found from the same table or Table III, supposing the curve to be of the degree $D_1 + \frac{1}{2}D$. Or,

$$BX = c' \sin \tfrac{1}{2}D_1, \quad \text{and} \quad Cc = c \sin (D_1 + \tfrac{1}{2}D).$$

Suppose $c' = 80, \quad D = 4°.$

$$\therefore \quad D_1 = 3° \, 12',$$

and $(D_1 + \frac{1}{2}D) = 5° \, 12'.$

Table II gives $BX = 2.23,$

and $Cc = 9.06.$

$$\therefore \quad CX' = 11.29.$$

Place B 80 feet from A and 2.23 feet from the tangent AT; then C a chain-length from B and 11.29 feet from the tangent AT.

Locate K from the tangent at L in the same manner as B is located. This gives the line CK.

Suppose the stations already located; the long chords EG, DH, etc., drawn, and also the perpendiculars Dd, Eel, etc.

Now $Cd = Cm - Di = \frac{1}{2}(CK - DH)$ = one half of the difference between the chords of six and of four stations. See Table IV.

$dl = Di - Ef = \frac{1}{2}(DH - EG)$ = one half of the difference between the chords of four and of two stations.

lm = one half the chord of two stations.

Again, $Dd = Fm - Fi$ = the difference between the middle ordinates of chords of six and of four stations. (See Table V.)

$El = Fm - Ff$ = the difference between the middle ordinates of chords of six and of two stations.

Finally, Fm is the middle ordinate of a chord of six stations.

We note that points and lines on the right of Fm are of course symmetrical with corresponding points and lines on the left.

Hence
$$Cn = CK - nK = CK - Cl$$
$$Ch = CK - hK = CK - Cd, \text{ etc.}$$

Now lay off Cd, Cl, Cm, etc.

Set D a station from C and a distance Dd from d, or CK; ther E a station from D and a distance El from l, or CK; then F station from E and a distance Fm from m, or CK, etc., etc.

We also have

$$Cd = c \cos \tfrac{5}{2}D, \quad dl = c \cos \tfrac{3}{2}D, \quad \text{and} \quad lm = c \cos \tfrac{1}{2}D;$$
$$Dd = c \sin \tfrac{5}{2}D, \quad Ee = c \sin \tfrac{3}{2}D, \quad \text{and} \quad Ff = c \sin \tfrac{1}{2}D.$$

These quantities may be taken directly from a table of sines and cosines, as already pointed out. It is not necessary, as shown under the preceding method, to compute and lay off Cd, Cl, etc.

D. *To Locate a Curve by Chord Offsets.*

Let $ABCDEF$ be the curve, having short chords $AB = c'$, subtending an angle $AOB = D_1$ and $EF = c''$, subtending an angle $EOF = D_2$.

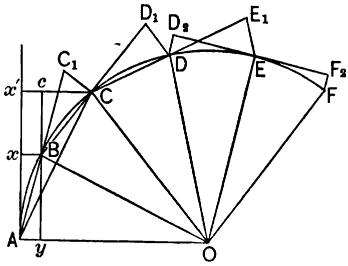


FIG. 22.

Locate B as in the las method. Prolong AB, making $BC_1 = c$. The angle CBC $= CAB + ACB = \tfrac{1}{2}D + \tfrac{1}{2}D_1$. Hence C_1C is the chord offse $2t'$ for a curve of $\tfrac{1}{2}D + \tfrac{1}{2}D$ degrees.

$$C_1C = 2c \sin \tfrac{1}{2}(\tfrac{1}{2}D + \tfrac{1}{2}D_1)$$
$$= 2c \sin \tfrac{1}{4}(D_1 + D).$$

Place C, therefore, at a distance c from B, and $2t'$ from C_1, $2t'$ being twice the tangent offset which is given by Table III opposite th degree $\frac{1}{2}(D + D_1)$. Prolong BC to D_1, making $CD_1 = c$. Now $D_1D = 2t$ is given by Table III, being twice the tangent offset. Place D, therefore, at a distance c from C, and $2t$ from D_1. Place all regular stations similarly. C may be placed, also, as in the preceding method. F is easily placed from the tangent at E, as B was placed from the tangent at A.

It may be observed that by the above method we are able, by means of a table giving offsets for full stations only, to locate any station, as C, by chord (or tangent) offsets, though the chord, as AB, preceding the adjacent chord may be of any length.

E. *To Locate a Curve by Middle Ordinates.*

Let $ABCDEFG$ be the curve, having a short chord $AB = c'$, subtending an angle $AOB = D_1$, and a short chord $FG = c''$, subtending an angle $FOG = D_2$.

Locate the stations B and C from the tangent Ax, or by some other method as already explained. Then set off on CO the distance $Cc = t = c \sin \frac{1}{2}D =$ the middle ordinate of a chord of two stations; and set D in the prolongation of Bc, and at a distance c from C. Set off Dd the same as Cc, and set E in the prolongation of Cd, and at a distance c from D. Locate all subsequent stations in the same way. To test the accuracy of the work, measure the perpendicular Fy from the last regular station upon the tangent at G. This distance ought to be

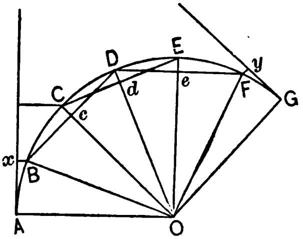


FIG. 23.

$$t_1 = c'' \sin \tfrac{1}{2}D_2.$$

F. *To Lay out a Curve by Radial Lines from the Center.*

Consider the case of a half-mile race-track, having two parallel sides, each 600 feet long, connected at the ends by semicircles, as shown in Fig. 24. Now $\frac{2640 - 1200}{2} = 720$, the length of each semicircle. Hence the degree of each curve $= \frac{180°}{7.20} = 25°$, and the radius is 229.18 feet.

Set the instrument at O, and run radial lines $O1$, $O2$, etc., making

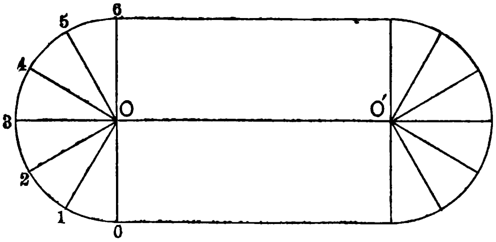


FIG. 21.

angles of 30° with each other, for example, and set stakes at 1, 2, etc., 229.18 feet from O. These stakes are $100 \frac{30}{25} = 120$ feet apart, measured on the arc.

Determine other points on the curve by middle ordinates.

The inside of the track is three feet inside of the line measured above.

ERRORS IN FIELD-WORK.

In laying out curves, as well as in all field operations, errors will sometimes occur, and it is important to know the immediate and the ultimate effect of such errors; to know when such errors are increasing, from one stage of the work to another, and when they are decreasing; when they are too small to be of importance, and when they are so large as to vitiate the result if not corrected.

It is important, also, to know the law of increase or of decrease of such errors; for in that case the error at any point may be found from that at any other point, the end of the curve, for example; thus making it possible to properly, that is, *really*, correct the curve without rerunning it.

I. *Curves Laid out by Deflection Angles.*

A. *To find an approximate value of the error at the end of a curve due to an error in the length of a chord.*—Suppose A, B, C, etc., to represent stations on the true curve, and A', B', C', etc., stations on the false curve.

1°. Suppose the first stake from A to be set at B', a distance $BB' = e$ beyond B, its true place.

Then C' will be set on AC prolonged and a chain from B', D' will be set on AD prolonged and a chain from C', etc.

To prove that CC' is less than BB'. Draw CH equal and parallel to BB'. Then $BB'CH$ is a parallelogram, and $B'H = BC = B'C'$ = a chain. Hence C' is on the arc whose center is B' and radius $B'H$. With C as center and radius $CH = e$ describe the arc HK. HK lies outside of the arc HC', since the arcs at H are perpendicular to CH and $B'H$ respectively, and are limited by AC prolonged. Hence

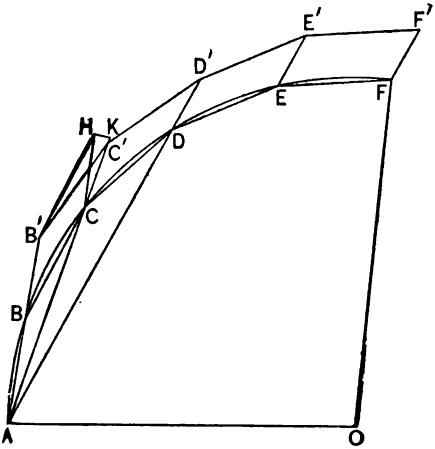


FIG. 25.

$$CC' < CK = CH = BB' = e.$$

For the same reason DD' is less than CC', and so on to the end of the *run*, that is, to the last stake set with the instrument at A.

Suppose the instrument moved to D'.

Since, in setting E' from D', we turn off from $D'A$ the same angle that we would turn off from DA in order to set E, we have $D'E'$ equal and parallel to DE, and therefore EE' is equal and parallel to DD'. For the same reason FF' is equal and parallel to EE' or DD'. So on to the end of the curve.

2°. Suppose an error to occur in some other chord than the first—in the second chord, for example. B is supposed, therefore, to be set correctly. Suppose the next stake set at C' instead of at C

Let $BC' = c + e$. Now $BC + CC' > BC'$, or

$$c + CC' > c + e. \qquad \therefore\ CC' > e.$$

From C' the errors will follow the same law as in the former case.

With reference to these two cases we remark that if the error occurs in the chord adjacent to the instrument, the errors in the stations will decrease slightly to the end of the run, and from that point remain constant to the end of the curve; but if the error occurs in a chord not adjacent to the instrument, the error in the station, at the end of that chord, is slightly greater than the error in the chord, though the errors slightly decrease from this point to the end of the run, and remain constant from the end of the run to the end of the curve. In all cases the error at the end of the curve may be regarded as equal to the error in the chord, whether adjacent to the instrument or not.

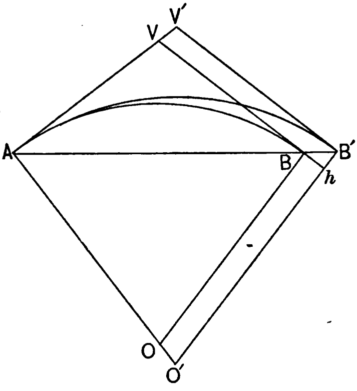


FIG. 26.

3°. Suppose the chain is in error, in which case *all* the chords will be in error.

Let the curve AB ($R = AO =$ radius, and $D =$ degree) be run with a chain 100 feet long, and the curve AB' ($R' = AO' =$ radius) with a chain $100 + e$ feet long. The number of chords is equal to $\frac{V}{D} = n$. Then for the difference in length of the curves we have $AB' = (100 + e)n$, $AB = 100n$, and therefore

$$AB' - AB = en. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

For the chord we find, since $R' = R\left(\frac{100 + e}{100}\right)$,

$$AB' = 2R\left(\frac{100 + e}{100}\right) \sin \tfrac{1}{2}V, \quad AB = 2R \sin \tfrac{1}{2}V,$$

and therefore

$$AB' - AB \doteq BB' = \frac{2Re \sin \frac{1}{2}V}{100}.$$

But $$R = 100\frac{180}{\pi D}.$$

$$\therefore \quad BB' = \frac{360e \sin \frac{1}{2}V}{\pi D} = \frac{114.60e \sin \frac{1}{2}V}{D}. \quad . \quad . \quad . \qquad (35)$$

Also

$$R' = (100 + e)\frac{180}{\pi D}, \quad \text{and} \quad R' - R = \frac{180e}{\pi D} = \frac{57.3e}{D}. \quad . \qquad (36)$$

We also have

$$B'h = BB' \sin B'Bh = BB' \sin \tfrac{1}{2}V = \frac{114.6e\,(\sin \frac{1}{2}V)^2}{D}, \quad . \qquad (37)$$

and $$Bh = BB' \cos \tfrac{1}{2}V = \frac{57.3e \sin V}{D}. \quad . \quad . \qquad (38)$$

Example.—Let $e = .02$, $V = 60°$, and $D = 6°$.

The error of the curve $= .02 \times \dfrac{60}{6} = .200$.

The error of the long chord $= \dfrac{114.6 \times \frac{1}{2} \times .02}{6} = .191$.

Also $$R' - R = \frac{57.3 \times .02}{6} = .191,$$

$$B'h = \frac{114.6 \times .02 \times \frac{1}{4}}{6} = .0955,$$

and $$Bh = \frac{57.3 \times .02 \times .886}{6} = .169.$$

B. Suppose that the first stake is set at B', Fig. 27, instead of at B, the error in the *angle* being $BAB' = A$.

Then it is evident that C', D', etc., will be set at the same angular distance from the chords AC, AD, etc., as B' is set from AB. B', C', etc., are on the curve, having O' as center and $AO' = AO$ as radius. $OAO' = BAB'$.

To find the error at the end of the curve.

Suppose E the end of the true curve.

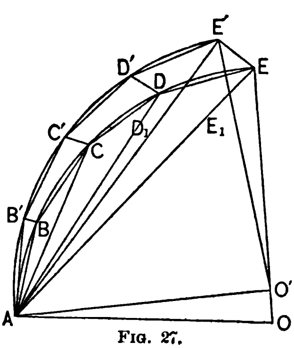


FIG. 27.

Now $AE' = AE$, and the angle EAE' = the angle BAB'.

Hence $EE' = 2AE \sin \frac{1}{2}EAE' = 2AE \sin \frac{1}{2}BAB'$.

If the error in the angle is corrected at the end of a run, say at E', then, for reasons given above, the error in the position of the stations is constant from E' onward to the end of the curve. If the error is corrected during a run, say just before D' is set, then that station will be set at D_1 on AD and 100 feet from C'. Similarly E_1 will be set on AE and 100 feet from D_1.

It may be shown that EE_1 is less than DD_1 precisely as it was shown that CC' is less than BB' in Fig. 25. Hence in this case the error will decrease to the end of the run, and remain constant from that point to the end of the curve.

II. *Curves Laid Out by Tangent Offsets.*

If in Fig. 22 the tangent at B is swung through an angle A, say, then all stations following B will be misplaced, the error increasing to the end of the "run" in the manner shown in Fig. 27. If a new tangent is drawn before reaching the end of the curve, as at E', Fig. 27, it is easy to see that such tangent would make an angle A with the tangent to the true curve at E, and that the error, therefore, would go on from E' forward precisely as from B to E', and hence the error would increase regularly from B to the end of the curve.

III. *Curves Laid Out by Ordinates from a Long Chord.*

In this case, if the end C of the chord CK, Fig. 21, is misplaced, then all stations from D to H inclusive will be misplaced in proportion to their *distances from K*; the greatest displacement being less than that of C. If K is also misplaced, the same stations will likewise share that error, in proportion to their *distances from C.*

IV. *Curves Laid Out by Chord Offsets.* (Fig. 22.)

If in this method any station, as B, is set at one side of its true position, then all subsequent stations will be in error in the same direction; the errors increasing regularly to the end of the curve, in the manner shown in Fig. 27.

If, owing to an error in some chord, some station is set forward or backward from its true position, then all subsequent stations error the same amount and in the same direction.

V. *Curves Located by Middle Ordinates.*

In Fig. 23 suppose C to be placed a distance a to the right, say, (that is, along CO,) of its true position. Then c will be a distance a to the right; D and d will be $2a$ to the right; E and e will be $3a$ to the right of their true positions, etc.

If C is correct, but c a distance a too far to the right, then D and d will be $2a$ to the right, E and e will be $4a$ to the right, F and f will be $6a$ to the right of their true positions, etc.

PROBLEMS IN SIMPLE CURVES.

I. Given the tangent distance $AB = d$ and the tangent offset $BD = t$, to find the radius of a curve that will pass through D and be tangent to AB at A.

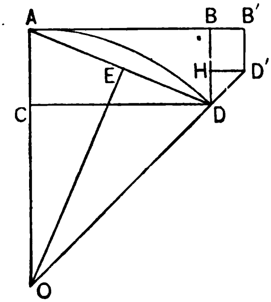


FIG. 28.

Let $AD = c$. We have

$$OD^2 = OC^2 + CD^2,$$

or

$$R^2 = (R - t)^2 + d^2.$$

From this we have

$$R = \tfrac{1}{2}\left(\frac{d^2}{t} + t\right). \quad . \quad . \quad (a)$$

From (a),

$$t = R - \sqrt{R^2 - d^2}, \quad . \quad . \quad (39)$$

and

$$d = \sqrt{t(2R - t)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Second Solution.—Let $AOD = A$; then

$$BAD = ADC = \tfrac{1}{2}A.$$

Now the triangle ABD gives

$$\frac{d}{t} = \cot \tfrac{1}{2}A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Also

$$\frac{t}{R} = \text{vers } A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and

$$\frac{R}{d} = \text{cosec } A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If d and t are given, find A from (41), then R from (42) or (43). If t and R are given, find A from (42), then d from (41) or (43). Finally, if d and R are given, find A from (43), then t from (41) or (42).

Example 1.—Given $t = 12$ and $d = 171.6$, to find R.

$$\tan \tfrac{1}{2}A = \frac{12}{171.6} = .06993.$$

$$\therefore\ \tfrac{1}{2}A = 4°, \quad \text{and} \quad A = 8°.$$

Then
$$R = \frac{t}{\text{vers } A} = \frac{12}{.00973} = 1233.3.$$

Example 2.—Given $R = 1233.3$ and $d = 171.6$, to find t.

$$\text{cosec } A = \frac{1233.3}{171.6} = 7.187. \quad \therefore\ A = 8°.$$

$$\text{Then } t = 1233.3 \times .00973 = 12.$$

This problem is useful in finding points on a curve beyond an obstacle.

II. To find the distance to a curve in a given direction from a given point on a tangent.

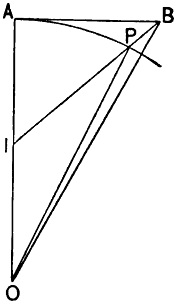


FIG. 29.

We have

$$AO = R, \quad AB = d, \quad \text{and} \quad ABP = B.$$

$$\tan ABO = \frac{R}{d}.\ PBO = ABO - ABP;$$

$$\sin OPI = \sin OPB = \frac{OB}{R} \times \sin PBO$$

$$= \frac{\sin PBO}{\sin ABO}.$$

Then

$$PB = R\frac{\sin (OPI - PBO)}{\sin PBO}. \quad . \quad (44)$$

This problem furnishes a general method of finding any desired point on a curve when obstacles preclude the usual methods. (See Problem 12.)

III. Having run the curve AD, radius $AO = R$, Fig. 28, to find the radus R' of a curve that will pass through D', given by angle $BDD' = D$, and $DD' = E$.

Let $AB = d$, and $BD = t$.

Draw $D'H$ parallel to AB. Now

$$HD' = E \sin D, \quad \text{and} \quad HD = E \cos D.$$

$$AB' = d + E \sin D = d_1, \quad \text{and} \quad B'D' = t - E \cos D = t_1.$$

Then $$R' = \frac{1}{2}\left(\frac{d_1^2}{t_1} + t_1\right).$$

It will be observed that when $AB' < AB$, $E \sin D$ must be subtracted from d to give d_1; and that when $B'D' = BH > BD$, $E \cos D$ must be added to t to give t_1.

IV. Having run a curve of radius R and tangent T, to find the new tangent T' corresponding to a new radius R', or to find a new radius R' corresponding to a new tangent T', the central angle remaining constant.

Eq. (15) gives

$$T' = R' \tan \tfrac{1}{2} V; \quad T = R \tan \tfrac{1}{2} V;$$

$$\therefore\ T' - T = (R' - R) \tan \tfrac{1}{2} V, \quad . \quad . \quad . \quad . \quad (45)$$

or $$R' - R = (T' - T) \cot \tfrac{1}{2} V. \quad . \quad . \quad . \quad . \quad (46)$$

Similarly, from eq. (17), we get

$$C' - C = 2(R' - R) \sin \tfrac{1}{2} V. \quad . \quad . \quad . \quad . \quad (47)$$

These equations are of advantage for computing the *change* in one element, $T' - T$ for example, from the *change* in another, $R' - R$ for example, when the given change is small, or is an aliquot part of the element changed.

Example 1.—Having run the curve of radius $R = 1910$, and the central angle $V = 46° \ 12'$, and tangent distance $T = R \tan \frac{1}{2} V$ $= 1910 \times .42654 = 814.7$, to find T' when R is made equal to 1900.

Eq. (45) gives

$$\begin{aligned} T' &= T + (R' - R) \tan \tfrac{1}{2} V \\ &= 814.7 - .42654 \times 10 = 814.7 - 4.27 = 810.4. \end{aligned}$$

Example 2.—Given $V = 52° \ 04'$, $R = 5730$, and $T = .48845 \times .5730 = 2798.8$, to find R' corresponding to $T' = T + \frac{T}{12}$.

We have

$$R' = R + \frac{R}{12} = 2798.8 + 233.2 = 3032.0.$$

V. Given a curve joining two tangents, to change the position of the P. C. so that with the same radius the curve may end in a given parallel tangent.

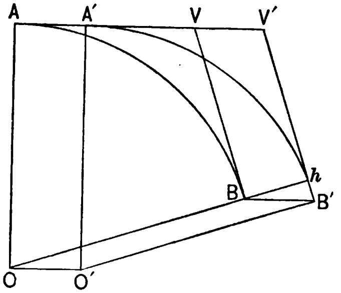


Fig. 30.

Let AB be the given curve, and $B'V'$ the parallel tangent. $VV' = a$ shows the distance and direction that all points of the curve are moved. The curve will therefore begin at A' and end at B'; AA' and BB', as well as OO', being equal and parallel to VV'. It is not necessary to run the tangent $B'V'$ in order to find the distance VV'. To find this distance run a line, such as BB', parallel to AV from any point on BV to meet $B'V'$. Then make $AA' = BB'$.

If the perpendicular offset $Bh = h$ is measured, we have

$$AA' = BB' = \frac{h}{\sin V},$$

V being the vertex angle.

If $B'V'$ were on the other side of BV from that shown, the new tangent point A' would fall on the opposite side of A from that shown in the figure.

If the new curve is required to end at a given point on $B'V'$, we have, then, the length of the new tangents,

$$A'V' = B'V' = T',$$

which gives the position of A' (and B'), and the corresponding radius,

$$R' = T' \cot \tfrac{1}{2} V,$$

or, by (46),

$$R' = R + (T' - T) \cot \tfrac{1}{2} V.$$

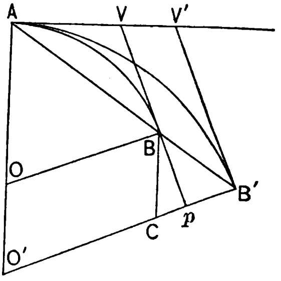


Fig. 31.

VI. Given a curve AB joining two tangents, to find the radius of a curve that from the same P. C. will end in a given parallel nt, Let $AV = BV = T$, $AV' = B'V' = T'$, $AO = R$, $AO' = R'$. We have, from the figure,

$$\frac{R'}{R} = \frac{T'}{T}, \quad \text{or} \quad R' = R\frac{T'}{T}.$$

Also, from eq. (46),

$$R' = R + (T' - T) \cot \tfrac{1}{2} V.$$

Or, prolong AB to B' and measure BB'. Let $AB = c$, and $AB' = c'$, $BB' = c' - c$. Then, from the figure or from (17),

$$R' = R + \frac{c' - c}{2 \sin \frac{1}{2} V}.$$

If the parallel tangent is defined by a perpendicular offset, as $B'p = h$, draw BC parallel to AO. Then

$$Cp = BC \cos BCp = (R' - R) \cos V.$$

$$\therefore\ CB' = (R' - R) = (R' - R) \cos V + h, \quad \text{or} \quad (R' - R)(1 - \cos V) = h,$$

$$\text{or} \quad (R' - R) \operatorname{versin} V = h, \quad \text{or} \quad R' = R + \frac{h}{\operatorname{versin} V}.$$

The quantity added to R in the above equations must be subtracted from it to find R' in the case in which V' falls between A and V, that is, when T' is less than T.

Example 1.—$V = 78°$, $R = 954.9$. T may be computed or found by Table VII to be 773.3.

Let $VV' = 20$ feet. Then

$$R' = R + (T' - T) \cot \tfrac{1}{2} V = 954.9 + 20 \times 1.2349 = 979.6.$$

Example 2.—$R = 1909.9$, $V = 46° 38'$. T may be computed or found by Table VII to be 823.2.

It is desirable to move the vertex from V to V' *about* 100 feet. Find the new radius R'.

$$823.2 \div 8 = 102.9;$$

$$1909.9 \div 8 = 238.7.$$

Hence $\qquad AV' = 823.2 + 102.90 = 926.1,$

and the new radius

$$AO' = R' = 1909.9 + 238.7 = 2148.6.$$

VII. Given a curve joining two tangents, to find the new tangent points, corresponding to the same radius, after each tangent has been moved any distance in the direction of the other.

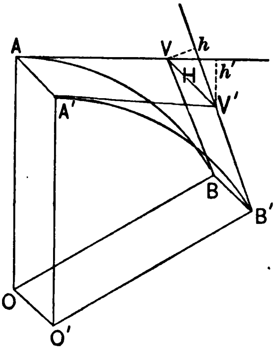


FIG. 32.

Let AV and BV be the given and $A'V'$ and $B'V'$ the required tangents. Let H be at the intersection of AV and $B'V'$. Let $VH = a$, $V'H = b$, and $VV' = c$.

We observe that

$$VHV' = 180^\circ - V.$$

$$\therefore \sin VHV' = \sin V,$$

and $$\cos VHV' = -\cos V.$$

Chap. III, formula No. (8), gives

$$\tan HVV' = \frac{\sin V}{\frac{HV}{HV'} + \cos V} = \frac{\sin V}{\frac{a}{b} + \cos V}.$$

Then $$VV' = \frac{b \sin V}{\sin HVV'}.$$

We observe that the directions of VH and HV' are the same as that in which the tangents BV and AV are moved, and therefore there can be no ambiguity about the direction of these lines or of VV', which is the line joining V and V'.

Since R and V are not changed, it is evident that all parts of the curve are moved in the direction VV' and a distance equal to VV'.

Hence make the angle $VAA' = HVV'$, and $AA' = VV'$.

The curve will begin at A' and end at B', BB' as well as OO' being equal and parallel to VV'.

If the distances the tangents are moved are given by perpendicular offsets $V'h' = h'$, and $Vh = h$, the triangles VHh and $V'Hh'$ are similar and give

$$\frac{HV}{HV'} = \frac{h}{h'}.$$

$$\therefore \tan HVV' = \frac{\sin VHV'}{\frac{HV}{HV'} - \cos VHV'},$$

or
$$\tan HVV' = \frac{\sin V}{\frac{h}{h'} + \cos V}.$$

Now the triangle hVV' gives
$$VV' = \frac{h'}{\sin HVV'}.$$

VV' and Vh are on the same side of BV; also, VV', and $V'h'$ are on the same side of AV.

VIII. Given a curve AB joining two tangents AV and BV, to change the curve so as to end at the same point as before, but in a tangent inclined at a given angle A with the original tangent.

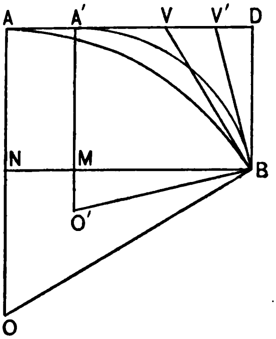


FIG. 33.

Let $AV = BV = T$, and the new tangents $A'V' = BV' = T'$.

Let $AO = R$, and $A'O' = R'$.

Draw $BD = p$ perpendicular, and BMN parallel to AV.

Let
$$V' = BV'D = BVD + VBV' = V + A.$$

Now $An = R$ versin V,

and $A'm = R'$ versin V'.

But $A'm = An$; $\therefore R$ versin $V = R'$ versin V',

or
$$R' = \frac{R \text{ versin } V}{\text{versin } V'}.$$

With this value of R' run the curve back from B through the angle V', and it will end at A', tangent to AV; $A'V'$ being equal to BV'.

If the length of the new tangent is desired, we have, from the triangle VBV',
$$T' = \frac{T \sin V}{\sin V'} = \frac{BD}{\sin V'} = \frac{p}{\sin V'}.$$

Then
$$R' = \frac{T'}{\tan \frac{1}{2}V'} = \frac{p}{\sin V' \tan \frac{1}{2}V'} = \frac{p}{\text{vers } V'}.$$

If we wish to run the curve from A, we have

$$DV = p \cot V, \quad \text{and} \quad DV' = p \cot V'.$$

$$\therefore\ VV' = p(\cot V - \cot V'),$$

and $$AA' = AV + VV' - A'V' = VV' + T - T'.$$

When $V < 90°$, T increases as V decreases, and *vice versa*; and when $V > 90°$, T and V increase and decrease together. In all cases R increases as V decreases, and *vice versa.*

IX. Given a curve, radius $AO = R$, joining the tangents AV and BV, to find the radius $AO' = R'$ of a new curve starting from A when the forward tangent VB' takes a new direction from the vertex.

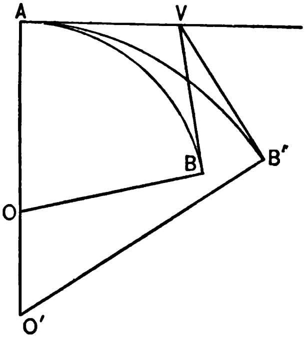


FIG. 34.

We have

$$VA = R \tan \tfrac{1}{2} V,$$

and

$$R' = VA \cot \tfrac{1}{2} V'.$$

$$\therefore\ R' = R \tan \tfrac{1}{2} V \cot \tfrac{1}{2} V'$$

$$= R \frac{\tan \frac{1}{2} V}{\tan \frac{1}{2} V'}.$$

X. Given a curve AB, radius $AO = R$, joining the tangents AV and VB, to find the change in the P.C., the radius remaining the same, when the forward tangent takes a new direction from the vertex.

We have

$$VA = R \tan \tfrac{1}{2} V;$$

$$VA' = R \tan \tfrac{1}{2} V'.$$

$$\therefore\ AA' = R(\tan \tfrac{1}{2} V - \tan \tfrac{1}{2} V').$$

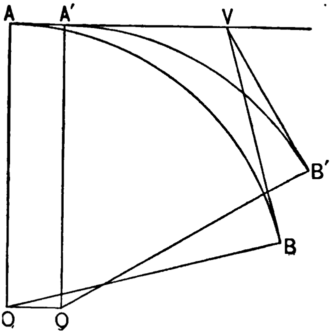


FIG. 35.

XI. Given the angle of intersection V of two tangents AV and BV, to find the radius

R and tangent distance T of a curve joining the tangents and passing through the point E.

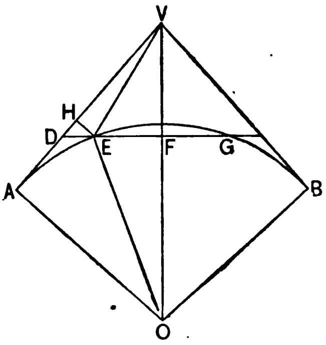


FIG. 36.

1°. Let E be given by $VE = l$, and angle $EVO = A$.

Let $VEO = E$, $VOE = O$, and $AO = R$, $AOV = \frac{1}{2}V$.

Now

$$\cos\tfrac{1}{2}V = \frac{AO}{VO} = \frac{EO}{VO} = \frac{\sin A}{\sin E};$$

$$\therefore\ \sin E = \frac{\sin A}{\cos\frac{1}{2}V}.$$

This gives E. Then

$$O = 180^\circ - (A + E).$$

Moreover,

$$\frac{EO}{EV} = \frac{R}{l} = \frac{\sin A}{\sin O}, \quad \text{or} \quad R = l\frac{\sin A}{\sin O}.$$

Finally, $T = R\tan\frac{1}{2}V$.

2°. If E is given by VH and HE perpendicular to each other,

then $$\tan EVH = \frac{EH}{VH};$$

$$EVF = FVH - EVH = 90^\circ - \tfrac{1}{2}V - EVH;$$

$$VE = \frac{VH}{\cos EVH}.$$

With these values proceed as above.

3°. Let E be given by $VD = a$, $DE = b$, the angle VDF being equal to $VOA = \frac{1}{2}V$. Produce DE to F and G.

Let $DF = c$. Then $c = a\cos\frac{1}{2}V$, and $AD = \sqrt{DE \times DG}$.

But $DG = DF + FG = DF + EF = 2DF - DE = 2c - b$.

$$\therefore\ AD = \sqrt{b(2c - b)}, \quad \text{and} \quad VA = VD + DA = T.$$

Now $R = T\cot\frac{1}{2}V$,

or $HE = DE\sin EDH = b\sin\frac{1}{2}V$.

$$DH = b \cos \tfrac{1}{2} V.$$

Then $$VH = VD - DH = a - b \cos \tfrac{1}{2} V.$$

With these values of VH and HE proceed as above.

XII. Given a tangent and curve (Fig. 36), to find the distance from a given point on the tangent to the curve in a given direction.

Let V be the point, and suppose the direction defined by the angle $EVA = B$. Let $AV = T$. We have

$$\tan \tfrac{1}{2} V = \frac{T}{R}.$$

Then $$EVF = DVF - EVA = (90^\circ - \tfrac{1}{2} V) - B = A, \text{ say.}$$

Now equation under Problem XI gives

$$\sin E = \frac{\sin A}{\cos \frac{1}{2} V}.$$

Then $$O = 180^\circ - (A + E),$$

and $$VE = l = R \frac{\sin O}{\sin A},$$

or,

$$\tan FVD = \frac{R}{T}, \quad \text{and} \quad EVF = FVD - EVD = FDV - B = A.$$

Then find sin E, then O and l as before.

Example.—$R = 954.93$, $T = AV = 350$, $AVE = 40^\circ$. We have

$$\tan AVO = \frac{R}{T} = 2.72837; \quad \therefore\ AVO = 69^\circ\ 52',$$

and $$AOV = \tfrac{1}{2} V = 20^\circ\ 8', \text{ and } EVF = 29^\circ\ 52' = V.$$

$$\sin E = \frac{\sin A}{\cos \frac{1}{2} V} = \frac{.49798}{.93889} = .53039; \quad \therefore\ E = 147^\circ\ 58';$$

$$O = 180^\circ - 177^\circ\ 50' = 2^\circ\ 10';$$

$$l = \frac{954.93 \times .03781}{.49798} = 72.4.$$

XIII. To locate a tangent to a curve of given radius R from a given point V. (Fig. 37.)

1. If the curve is marked by stakes visible from the given point, a tangent can be sighted in at once.

2. If the curve is not visible from the point, run a trial tangent VB, and measure $VB = A$ and the angle $VBO = B$.

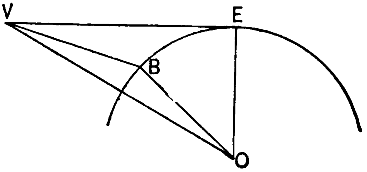


FIG. 37.

Chapter III, formula (8), gives

$$\tan BVO = \frac{\sin B}{\frac{A}{R} - \cos B}.$$

Then $$OV = OB \cdot \frac{\sin B}{\sin BVO}, \quad \text{and} \quad \sin EVO = \frac{EO}{OV}.$$

Lastly, $$BVE = EVO - BVO.$$

This gives the angle to be laid off from the trial tangent to give the true tangent AE.

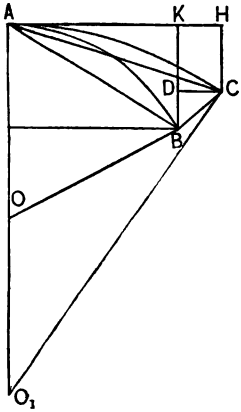


FIG. 38.

XIV. Given two curves AB and AC, radii $AO = R$, $AO_1 = R_1$, subtending the central angles $AOB = O$ and $AO_1C = O_1$, to find the length of the line BC. (Fig. 38.)

Let A, B, and C represent the angles of the triangle ABC, and a, b, and c the sides opposite. We have

$$c = 2R \sin \tfrac{1}{2}O, \quad \text{and} \quad b = 2R_1 \sin \tfrac{1}{2}O_1.$$

Also,

$$BAO = 90^\circ - \tfrac{1}{2}O, \quad CAO_1 = 90^\circ - \tfrac{1}{2}O_1$$

$$\therefore\ BAC = A = \tfrac{1}{2}(O - O_1).$$

Now $$\tan B = \frac{\sin A}{\frac{c}{b} - \cos A}, \text{ (See Chap. III, formula (8).)}$$

and then $$a = b\frac{\sin A}{\sin B}.$$

If the curves are run through an integral number of stations, Tables IV and V give at once the tangent distances AK and AH and the tangent offsets BK and CH. Then, drawing CD parallel to AH to meet BK in D, we have

$$CD = AH - AK = d, \text{ say, and } BD = BK - CH = t, \text{ say.}$$

Then $$BC = \sqrt{d^2 + t^2},$$

or $$\tan CBD = \frac{CD}{BD} = \frac{d}{t},$$

and $$BC = \frac{BD}{\cos CBD}.$$

OBSTACLES IN SURVEYING.

It is often necessary to draw lines parallel and perpendicular to other lines. Hence the following problems :

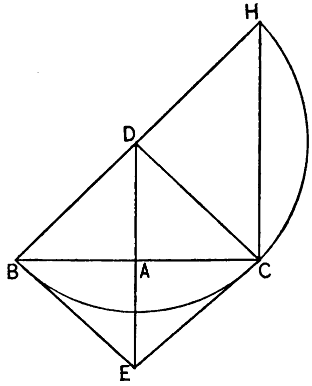


FIG. 39.

I. *To erect a perpendicular at any point of a line.* (Fig. 39.)

1. Let A be the point, and BC the line. Make $AB = AC$, and with B and C as centers and any radius greater than AB describe arcs intersecting at D or at E, or (with a different radius) at F. Any two of the points A, D, E, and F determine the perpendicular required.

2. Fix any two points of the chain at B and at C. Take hold of the point midway between B and C and stretch the chain, the middle point being at D. AD is the perpendicular required. We may find, similarly, other points E, F, etc. Any two of these points A, D, E, F, etc., determine the perpendicular required.

3. Let C be the point. Take any point D as a center, and with a radius DC describe an arc BC. Prolong BD, making $DH = BD$. CH is the perpendicular required. For DA (A being at the middle of BC) is perpendicular to BC, and, by construction, CH is parallel to AD.

4. A right angle may be obtained by laying off on the ground the three sides of any of the triangles represented in the following ble, or any equimultiples of these sides, making one of the

sides adjacent to the right angle (a or b) coincide with the line. Let c = the hypothenuse.

No.	*a.*	*b.*	*c.*	No.	*a.*	*b.*	*c.*
1	3	4	5	6	20	21	29
2	5	12	13	7	12	35	37
3	8	15	17	8	9	40	41
4	7	24	25	9	11	60	61
5	10	24	26	10	13	84	85

Thus, in Fig. 40, using 70 links of the chain, hold the first end, also the end of the 70th link of the chain, at A, the end of the 21st link at B, and the end of the 50th link at C.

If in the three expressions $m^2 - n^2$, $2mn$, and $m^2 + n^2$ we assign to m and n any values at pleasure, m being greater than n, we will have sets of numbers representing the sides of right-angled triangles. In that way the above numbers were found. Equimultiples of any set of the above numbers will represent the sides of a right-angled triangle.

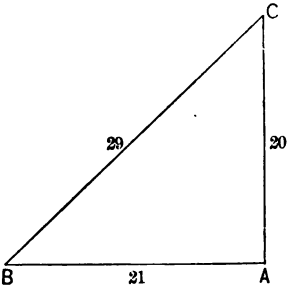


FIG. 40.

II. *To let fall a perpendicular from a given point to a given line.*—Let H in Fig. 39 be the point. Measure any line HB to the given line. At the middle of HB take D as a center, and with a radius DB describe an arc BC. HC is the required perpendicular. For continuing the arc to H, we see that the angle BCH is inscribed in a semicircle.

III. *To let fall a perpendicular to a line from an inaccessible point.*—Let BC (Fig. 41) be the line, and P the point. Let p represent the perpendicular PK. Then

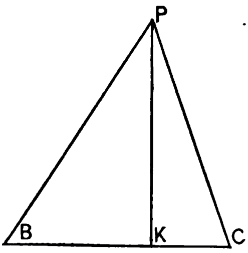


FIG. 41.

$$BK = p \cot B, \text{ and } CK = p \cot C.$$

$$\therefore \frac{BK}{CK} = \frac{\cot B}{\cot C},$$

and
$$\frac{BK}{BK + CK} = \frac{\cot B}{\cot B + \cot C}.$$

Since $BK + CK = BC$, we have

$$BK = BC\frac{\cot B}{\cot B + \cot C}.$$

This gives the foot of the perpendicular K. If BC is taken equal to 100 or some small multiple of 100, BK is very easily found.

This problem is particularly useful in locating important objects, such as mills, warehouses, bridges, etc., on one side or the other of a railway survey.

In this case B and C represent stations or points on the survey, and the angles at B and C can be measured and recorded while the instrument is set at B and at C. The simple division required to find the position of K can be made at any time. Of course the point P is located graphically by drawing BP and CP.

IV. *To prolong a line AB* (Fig. 42) *past an obstacle and to measure its length.*—This is easily done by perpendicular offsets, a method too familiar to need description, but not the best way.

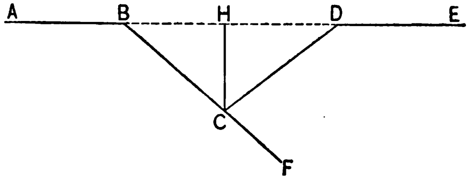


FIG. 42.

Or, measure BC in any convenient direction, and at C deflect any angle FCD. Draw BD and the perpendicular CH. The angle $CDB = FCD - CBD$. Hence

$$CD = BC\frac{\sin CBD}{\sin BDC};$$

also

$$BD = BC\frac{\sin FCD}{\sin BDC}.$$

If the angle BCD is made equal to 90°, then

$$CD = BC \tan CBD,$$

and

$$BD = BC \div \cos CBD.$$

If the angle FCD is made equal to $2CBD$, then

$$CDB = FCD - CBD = CBD.$$

Hence $CD = BC$, and $BD = 2BH = 2BC \cos CBH$.

CH is the departure of the line BC, or of DC, from the line ABD. It is also the approach of the line CB, or of CD, to ABD.

If necessary more than one course may be run away from the main line ABD, and more than one in returning to it.

To recover the main line it is only necessary to make the sum of the approaches equal to the sum of the departures.

The distance measured on the main line is obtained as above.

V. *Obstacles to Measurement.*—Methods have been pointed out in connection with Fig. 42 for finding the length of obstructed lines when the ends are accessible. When inaccessible the following problems apply.

A. When one end of the line is inaccessible. (Fig. 43.)

1. Let AB be the line to be measured, across a river for example. Measure AC in any convenient direction, and the angles at A and C. Then

$$AB = \frac{AC \sin C}{\sin B}$$

$$= AC \frac{\sin C}{\sin (A + C)}.$$

2. If the angle ACB is made equal to half of DAC, then

$$CBA = CAD - BCA$$

$$= 2BCA - BCA = BCA.$$

$$\therefore AB = AC.$$

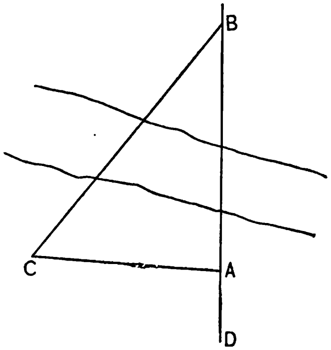


FIG. 43.

3. Or, in Fig. 43, make the angle $BAC = 90°$. Then

$$AB = AC \tan ACB.$$

If, in this, $AC = 100$, or some simple multiple of 100, which is usually easy to effect, the formula requires no computation whatever.

4. If ACB in Fig. 44 is made equal to 45°, $AB = AC$.

5. If at C we make the angles ACB and ACD equal, we have

$$AB = AD.$$

When the river or other obstruction occurs on a continuous survey, as a railway survey, AD is a measured line, and this method gives $AB = AD$ without any computation whatever.

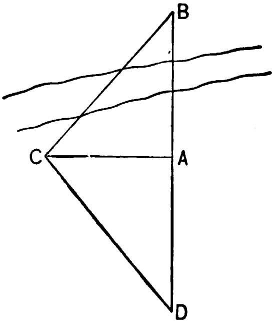


FIG. 44.

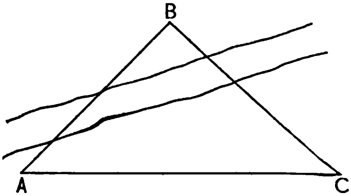


FIG. 45.

6. In Fig. 45, AB being the distance required, run and measure any line AC and measure the angle $BAC = A$. Make

$$ACB = 90° - A = C.$$

Then $$AB = AC \sin C.$$

B. When both ends of the line are inaccessible. (Fig. 46.) Let AB be the line to be measured. Find the distances from the point C to each end of the line A and B by preceding methods, and measure the angle C. Then

$$\tan A = \frac{\sin C}{\frac{AC}{BC} - \cos C};$$

(see Chapter III, formula (8).)

$$AB = BC\frac{\sin C}{\sin A}.$$

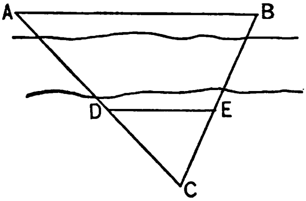


FIG. 46.

C. To erect, at a given point A (Fig. 47), a line AK perpendicular to an inaccessible line BC, and to draw a parallel AH to the same line.—Find $AB = c$ and $AC = b$ by preceding methods. Then

$$\tan B = \frac{\sin A}{\frac{c}{b} - \cos A}.$$

Now draw AK, making

$$BAK = 90^\circ - B.$$

AK will be the required perpendicular, and AH, making $BAH = B$, will be the required parallel.

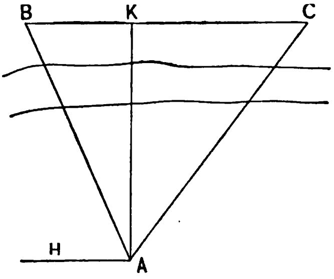


FIG. 47.

D. To find the length and relative position of an inaccessible line, AB, from an accessible line, CD, separated from the former by an inaccessible space.

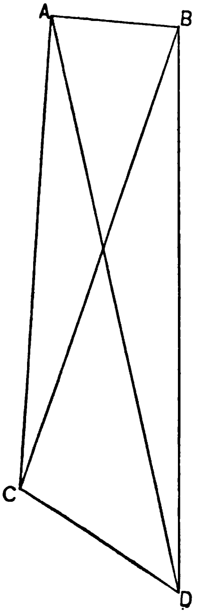


FIG. 48.

Measure CD and the angles at C and D.

Example.—Let $CD = 4000$;

$$BCD = 126^\circ\ 25\tfrac{1}{2}'; \quad ADC = 47^\circ\ 53\tfrac{1}{2}';$$

$$ACB = 3^\circ\ 10'; \quad ADB = 3^\circ\ 01'.$$

$$\therefore\ ACD = 129^\circ\ 35\tfrac{1}{2}'; \quad BDC = 50^\circ\ 54\tfrac{1}{2}'.$$

Also,

$$CAD = 180^\circ - ACD - ADC = 2^\circ\ 31';$$

$$CBD = 180^\circ - BCD - BDC = 2^\circ\ 40'.$$

Hence

$$AC = 4000 \frac{\sin 47^\circ\ 53\tfrac{1}{2}'}{\sin 2^\circ\ 31'} = 67581.9;$$

$$BC = 4000 \frac{\sin 50^\circ\ 54\tfrac{1}{2}'}{\sin 2^\circ\ 40'} = 66728.8;$$

$$\tan CAB = \frac{\sin C}{\frac{AC}{BC} - \cos C} = \frac{.0552406}{.0143115} = 3.85987.$$

$$\therefore\ CAB = 75^\circ\ 28\tfrac{1}{2}'.$$

Finally, $AB = BC \dfrac{\sin 3^\circ\ 10'}{\sin 75^\circ\ 28\tfrac{1}{2}'} = 3807.8.$

If AB is accessible, it can be measured as a check on the computation. Such is the case when it is a tangent of a railway survey adjacent to the inaccessible space.

The data of this example are taken from an actual night survey across an inaccessible sea-marsh.

Rockets were thrown and lights then exhibited at A and B. which were observed with transits from the tops of towers at C and D.

The computed and the measured length of AB agreed within a few inches.

The best method of making a preliminary railway survey through a wooded region is by a suitable adaptation of the method of traversing, which we will now explain.

So far as known to the author, this was first devised by him in 1869, and used for him by his assistant, Prof. J. B. Davis (now of Michigan University) in making the preliminary surveys of the Owosso and Northwestern Railway.

Suppose we wish to run from A in the direction of ABF, which we will call the base line; and upon which numerous obstacles, such as trees, occur, making it necessary to run the line $AB_1C_1D_1E_1F$, called a traverse. The deflection angles at B_1, C_1, D_1, and E_1 are supposed to be small.

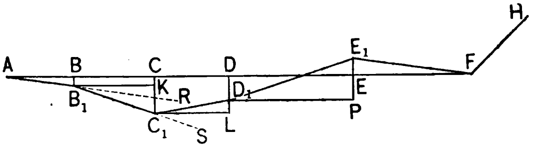


FIG. 49.

From B_1, C_1, D_1, and E_1 draw perpendiculars to AB, and from B_1, C_1, and D_1 draw parallels B_1K, C_1L, and D_1P to AB as shown. Prolong AB_1 to R, and B_1C_1 to S.

The course of a line is its direction with reference to the base line.

The departure of a line is the distance that a point recedes from or approaches to the base line in moving from one end of the line to the other.

We have $D_1L = C_1D_1 \sin D_1C_1L$. Now since the sines of small angles vary nearly with the angles or with the number of minutes in the angles, we see that the departure of a line varies nearly as the product of the length of the line by the number of minutes in the course.

The departure of a point is its distance from the base line. Thus the departure of $D_1 = DD_1$.

The departure of the end of a line, as B_1C_1, inclining from the base, is equal to the departure of the beginning of the line plus the departure of the line. Thus $CC_1 = BB_1 + C_1K$. The departure of the end of a line, as C_1D_1, inclining toward the base, is equal to the departure of the beginning of the line minus the departure of the line. Thus $DD_1 = CC_1 - D_1L$. The departure of the end of the line D_1E_1 which crosses the base line is equal to the departure of the line minus the departure of the beginning of the line. Thus $EE_1 = E_1P - DD_1$.

The record of the survey can be conveniently kept, as shown in the following table, the columns of the transit-book serving the purpose perfectly.

Stations.	Angles turned.		Angles with Main Line.		Departures.			
					Each Course.		Total.	
	Left.	Right.	Left.	Right.	Left.	Right.	Left.	Right.
$A =$ 10		10′		10′				
11								
12								
$B_1 =$ 13		22′		32′		3000		3000
14								
15								
$C_1 = +$ 20	42′		10′			7040		10040
16								
17								
18								
$D_1 =$ 19	30′		40′		3800			6240
20								
21								
22								
23								
$E_1 =$ 24		1° 20′		40	20000		13760	
25								
26								
27								
$F = +$ 44						13760	.00	.00

The deflection at station 10 is 10′ R., and at station 13 it is 22′ R., etc. The course from 10 to 13 is evidently 10′ R.; from 13 to 15 + 20 it is 10 + 22 = 32′ R.; from 15 + 20 to 19 it is 42 − 32 = 10′ L., etc.

From 10 to 13 the departure is $300 \times 10 = 3000$ foot-minutes.
" 13 " 15+20 the departure is $220 \times 32 = 7040$ "
" 15+20 to 19 " " " $380 \times 10 = 3800$ "
" 19 to 24 " " " $500 \times 40 = 20000$ " etc.

The aggregate departures are:

At 13.................................. 3000 R.
At 15 + 20......3000 + 7040 = 10040 R.
At 19............3000 + 7040 − 3800 = 6240 R., etc.

The distance necessary to run from a given station on any given course to reach the base line is found by dividing the departure at that station by the course. Thus from station 24 forward the course is 40′ R. Then 13760 ÷ 40 = 344 feet, showing that the auxiliary line (E_1F in the figure) will reach the base line 344 feet beyond station 24, or at 27 + 44.

The figure represents a main angle at *F*, the forward tangent being *FH*, and which may be followed approximately the same as *AF* was followed.

Let l = the length and d = the departure of any line, and n = the number of minutes in the course. Then

$$d = l \sin n' = ln \sin 1' = .0002909ln.$$

Since ln is given in the last two columns, the departures in feet are found by multiplying the quantities in these columns by .0002909 or .00029 nearly. We observe that

$$1 \div .0002909 = 3438 \text{ nearly.}$$

Hence the departures given in the table (in foot-minutes) divided by 3438 will give the departures in feet.

A rough approximation for the purpose of keeping sufficiently near the base line on sideling ground is generally all that is needed. This being the case, it is not in general necessary or advisable to find the total departures, except when it is desirable to "run for the base" preparatory to turning a main angle.

Thus to find the departure at station 24. The sum of the product to the left is.................. 3800 + 20000 = 23800
and the same to the right is........... 3000 + 7040 = 10040

The difference is.................................. 13760 L.

We have $$AB = AB_1 \cos BAB_1,$$
or
$$AB_1 - AB = AB(1 - \cos BAB_1) = AB \text{ vers } BAB_1.$$

For $BAB_1 = 2° 34'$ this becomes $AB_1 - AB = .001AB$ nearly. This shows that AB_1 exceeds the true distance measured along the base by only one thousandth part of its length for an angle of 2° 34′. If greater accuracy than this is desired, the angles between the auxiliary line and the base line, or the "courses," may usually be made smaller than 2° 34′. Since the error is approximately as the square of the number of minutes in the angle,

for 1° 17′ it is nearly .00025, or nearly 1 in 4000;
and for 0° 38½′ " " " .00006, " " 1 " 16000, etc.

To find the angle in minutes between the base line and a line joining any two stations.

Divide the difference or the sum of their departures, according as they are on the same or on opposite sides of the base line, by their distance apart.

Thus the line B_1D_1 makes with the base line the angle

$$\frac{DD_1 - BB_1}{BD} = \frac{6240-3000}{600} = 5'.4 = 5'\,24''.$$

The line AE_1 makes with the base an angle

$$\frac{EE_1}{AE} = \frac{13760}{1400} = 9'.83 = 9'\,50''.$$

In platting, the auxiliary lines are penciled only, so as to plat observed objects in proximity to the line necessarily observed from the auxiliary lines. When these objects and the base lines are mapped the auxiliary lines need not be retained.

This method yields quite accurate results when the angles between the lines of the survey are 2° or 3°, as we have seen.

For perfect accuracy, however, use the following method:

Problem.—Having run a broken line $ABCD$, to find the angle between the first course and the direct course AD. Represent the lines run, in their order, by a, b, and c, and the deflection angles at B and at C by B and C respectively. Draw CR parallel to AB, and CH and RDK perpendicular to AB.

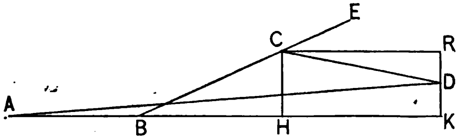


FIG. 50.

$$\text{Angle } DCR = DCE - RCE = C - B.$$

Now

$$BH = b\cos B;\quad HK = c\cos(C - B);\quad AK = AB + BH + HK.$$
$$CH = b\sin B;\quad DR = c\sin(C - B);\quad DK = CH - DR.$$

Now $$\tan DAK = \frac{DK}{AK}.$$

Also $$AD = AK \div \cos DAK.$$

Of course the method is applicable whatever the number of lines run. All but the last line could usually be taken equal to a whole number of chains, which would reduce the required computation to a simple multiplication.

E. To find the angle of deflection, V, between two straight lines AV and VM, when the point of intersection is inaccessible; and the distances of the intersection from given points on the lines.

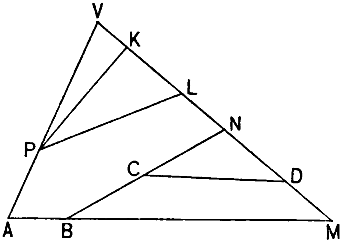


FIG. 51.

1. Run and measure a perpendicular PK to one of the lines. Measure the angle $VPK = P$. Then

$$V = 90^\circ + P;$$
$$KV = KP\tan P,$$
and
$$PV = KP \div \cos P.$$

2. Run and measure any line PL from one line to the other. Measure also the angles $VPL = P'$ and $PLV = L$. Then

$$V = P' + L.$$

Hence, in the triangle PVL, PL and the angles are known, to find VP and VL.

3. If obstructions prevent the use of the former methods, run and measure any broken line $ABCD$. Prolong AB and BC to meet VM at M and N.

Measure the deflection angles $CBM = B$, $DCN = C$, and $CDN = D$.

Let $AMV = M$, and $CNV = N$. Then

$$N = C + D;\quad CN = CD\frac{\sin D}{\sin N} = CD\frac{\sin D}{\sin(C + D)};$$

$$BN = BC + CN.\quad \text{Also}\quad M = N - B = C + D - B.$$

$$BM = BN\frac{\sin N}{\sin M}; \qquad AM = AB + BM.$$

Now we have AM and the angles at A and M, to find AV, MV, and the angle $V = A + M$. A similar explanation will apply to any case.

When a broken line must be used, the above method involves fewer computations than any other.

F. To locate a curve joining two tangents when the vertex is inaccessible.

Find by the last problem the distances Va and Vb to convenient points on the tangents, and the angle V. Then assigning or computing the tangent $T = AV$ or BV from the radius, we have

$$aA = T - aV,$$

and $$bB = T - bV.$$

We now have the tangent points and can run in the curve as usual.

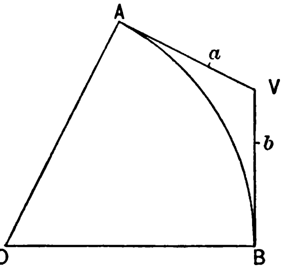


Fig. 52.

G. To locate a curve of radius R or tangent T when the vertex, the beginning, and the end of the curve are inaccessible. Find, as shown with Fig. 52, the angle V and the distance Va to any point, a, on AV. Then

$$aA = T - aV,$$

and

$$\sin AOc = \frac{cd}{cO} = \frac{aA}{R}.$$

Also

$$ac = Ad = R \operatorname{versin} AOc.$$

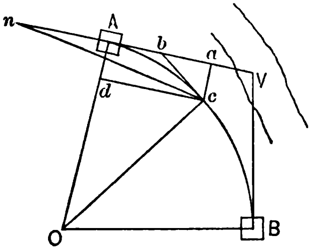


Fig. 53.

Drawing the tangent cb, we have

$$abc = AOc, \quad \text{or} \quad acb = 90° - AOc.$$

This gives the direction of the curve at c, and it may be run in each way from c.

To pass from any point c on the curve to any point n on the tangent.

We have $$\tan (dcn = anc) = \frac{ac}{an}.$$

Set the instrument at c and turn off from the tangent cb an angle

$$bcn = bcd - dcn$$
$$= abc - anc,$$

and measure $$cn = \frac{an}{\cos anc}.$$

H. To find any desired point on a curve when obstacles preclude the use of ordinary methods.

(1.) In Fig. 28 measure any convenient tangent distance $AB = d$. Then, as shown in Problem 1, eq. (43),

$$\operatorname{cosec} A = \frac{R}{d}; \quad \text{then} \quad t = d \tan \tfrac{1}{2}A.$$

d and t give the point D on the curve.

It is important to note that if AB is made equal to one half the long chord for any number of stations given by Table IV, $BD = AC$ is the corresponding middle ordinate and may be found in Table V.

Example.—Let AD be a 4° curve, and AB = one half the chord of four stations $= \frac{398.70}{2} = 199.35$. Then Table V gives

$$BD = 13.94.$$

(2.) Problems 2 and 12 of this chapter furnish general methods of overcoming obstacles on curves.

(3.) We can find points on the curve as follows:

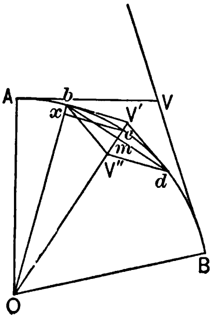


FIG. 54.

Let b be a station near the obstacle. Deflect from the tangent at b some small multiple of the deflection angle for one station $\frac{1}{2}D$, giving the line bd. The length of bd may be taken at once from Table IV and measured off, giving d, a station on the curve beyond the obstacle. Taking $bm = \frac{1}{2}bd$, and measuring off the middle ordinate mc taken from Table V, gives also a point c on the curve.

If more convenient, make $bx = cm$, and $xc = bm$, which also gives c.

Again, run the tangent $bV' = d'$ any convenient distance. Then

$$\cot bOV' = \frac{bO}{bV'} = \frac{R}{bV'} = \frac{R}{d'}.$$

Deflect at V' an angle equal to $2bOV'$, and make $V'd = bV'$. d will be a point on the curve. The number of stations from b is equal to

$$\frac{bOd}{D} = \frac{2bOV'}{D}.$$

bV' should usually be taken equal to a whole number of chains, in which case $\frac{R}{bV'}$ is very readily found.

The lines bV'' and $V''d$ lying on the inside of bd may be run instead of bV' and $V'd$.

CHAPTER V.

LEVELING, STADIA MEASUREMENTS, ETC.

THE field operations in connection with the level are more simple than those required with the transit, but they require greater skill and facility in manipulation in order to produce correct results.

It is to be observed that the elevation of points is a *relative* matter. The elevation of some point, from which *all others* are to be found, is arbitrarily assumed to be 100, or some other number sufficiently large, so that the elevation of all points to be considered will be greater than zero.

Near the coast, and in fact wherever practicable, it is important to refer the levels to the mean level of the sea, calling this zero, or 100, or some other number, taking care to establish from it some convenient and permanent reference-point called a bench-mark, or bench.

All points having the same height as this bench are sometimes said to be on a level surface called the datum. This, however, makes no difference with the work and serves no useful purpose, and need not be considered.

All elevations thus found become of much importance in determining the relative elevations of the country, and in the construction of physiographical maps, etc.

Having established the first bench, and recorded its elevation, the rodman stands squarely on both feet behind the rod, and rests it on the bench as nearly in a vertical position as possible, which is best done by simply steadying it with the thumbs and fingers, taking care not to grasp it.

The levelman sets up his level, preferably in the direction that the line extends, in any position from which he can well see the bench, as well as points to be afterwards observed.

He then makes sure that the instrument is in adjustment, and is focused; levels it carefully and sights to the rod. He may plumb the rod laterally by means of the vertical cross-wire of the level, and the rod may be waved gently on each side of the vertical toward and from the instrument, the shortest reading being the true reading.

The line of sight on the rod covered by the horizontal cross-wire is then on a level with, or at the same height as, the wire itself, and the latter is therefore higher than the bench by the distance intercepted on the rod between the line of sight and the bottom of the rod. This is called the reading of the rod, or simply the reading. Adding this reading to the height of the bench, we obtain the height of the cross-wire, technically called the height of instrument, and designated by the initials H. I.

Having obtained the height of instrument, the elevation of any other point upon which the rod can be read can be found by taking a reading of the rod upon it. Of course the point is below the instrument an amount equal to the reading, which must therefore be subtracted from the height of instrument to give the elevation of the point. The elevations of any number of points may be thus obtained.

In order to obtain the elevation of points above the instrument, or below it more than the length of the rod, the instrument must be moved from its present position to one higher or lower as the case may require.

Before the instrument is moved to a new position a *temporary bench*, called a turning-point (and designated by T. P. or "Peg"), must be established and its elevation ascertained with care, since any error in its elevation is carried forward throughout the whole line of levels. A turning-point must be firm and definite and not easily disturbed or lost. A small stake or "peg" driven with its upper surface about flush with the surface of the ground is generally used. The top of a rock may well serve the purpose.

Benches and turning-points are of course the same in principle, but the more or less permanent point taken as the basis of the elevations of the Survey, and also those made usually along and near the line, for future reference, whether used as turning-points or not, are usually called benches.

From this new turning-point we proceed precisely as before, by getting a new height of instrument, etc., and it is important to note that the operation just described, of obtaining a height of instrument from a bench or turning-point, and *then* obtaining the heights of any number of desired points within range of the instrument, including a new bench or turning-point, includes the whole subject of leveling.

Since the cross-wires must be higher than any point upon which a reading is taken it must be remembered that:

1. The reading on a point, added to its elevation, gives the height of instrument.
2. The reading on a point subtracted from the height of instrument gives the elevation of the point.

In other words: We must add a reading (to the height of some point) to get a height of instrument, and must subtract a reading (from a height of instrument) to get the height or elevation of some point.

The theory of leveling requires, therefore, only a simple application of addition and subtraction, and it is not easy, it would seem, to go wrong in it.

Station.	+ *S*	H. I.	− *S*	Elevs.	Remarks.
BM				200.00	W. Oak 60 ft. R. of Station 0
0	3.46	203.46	7.29	196.17	
1			5.34	198.12	
Peg 2	....	....	0.81	202.65	
+ 40	4.17	206.82	1.12	205.70	
3			3.16	203.66	
4			6.09	200.78	
5			4.14	202.68	
Peg + 60			1.07	205.75	
6	6.18	211.93	3.13	208.80	

The accompanying table shows a convenient form of field-book for keeping the level notes of a railway or other survey. The *first* column contains the stations and benches. The second the plus readings taken on points whose elevations are assumed or already determined. The third column contains the heights of instrument recorded one line below the elevation of the turning-point (or bench) from which it is calculated. The fourth column contains the minus readings. The fifth column contains the elevations of all points observed. The right-hand page is reserved for remarks describing the benches and their location, also objects crossed by (or near) the line, as roads, streams, ditches, etc.

It is to be observed that for any series of levels the sum of the plus sights less the sum of the minus sights (omitting those for determining intermediate points on the ground) is equal to the difference between the first and last elevation.

Thus to prove station 3, we have

$$3.46 + 4.17 - 0.81 - 3.16 = 203.66 - 200 = 3.66.$$

To prove the H. I., 211.93, we find

$$3.46 + 4.17 + 6.18 - 0.81 - 1.07 = 211.93 - 200 = 11.93.$$

In practice it is best to check each page of the field-book by comparing, as above, the first turning-point or height of instrument (brought over from the preceding page), with the last turning-point or height of instrument on the page.

To facilitate this work some engineers use two columns for the minus sights, placing those which determine the turning-points in a column by themselves.

This practice is commendable.

Benches should be established at short distances apart along the line, taking care to locate them, so far as possible, near the crossings of roads, streams, railways, etc., and at all points where their need can be foreseen, in the location of cattle-guards, culverts, bridges, etc. Of course an extra-good bench should be established at the end of the survey. An extra-good turning-point or bench should also be established at the end of each day's work.

The object of obtaining a line of levels is to furnish a profile of the line surveyed, showing the undulations of the surface over which it passes.

The elevations are platted on profile paper, the horizontal scale being about 400 feet to an inch, and the vertical scale about 25 feet to an inch. This distortion of scale magnifies the vertical measures about $^{400}/_{25} = 16$ times, so that the slight changes in the elevation of the surface may be distinctly seen.

In running a line of "flying" levels no readings are taken except on turning-points. If the difference of levels of the extreme points only is desired, it is necessary to find the difference only between the sum of the plus and of the minus readings, as already explained. This is very convenient for testing a line of levels already run; in which case it is best to touch on the benches only, and if found correct, the intermediate elevations may be regarded as correct also.

No line of levels should be taken as correct, and so used, without first being carefully checked.

The Philadelphia rod is the most convenient and best rod in use. It is plainly lettered and easy to use, and may be read by the levelman when desirable, and at a distance of several hundred feet.

To Locate a Level Line.—Set a peg at the desired height, as a starting-point, and take a reading of the rod thereon. Send the rod forward in the desired direction, and have it moved upward or downward along the slope of the ground until a point is found which gives the same reading as before.

Of course the reading is taken on a peg. This second peg is of the same height as the first. Find in the same way a third peg from the second, etc.

In this way stakes may be set at points on the ground level with the top of a proposed dam, or with the supposed top of water flowing over the dam. Then joining these stakes by lines,—the area thus inclosed may be measured.

The water behind a dam is not level, but is curved concavely upward and so increases in height back of the dam, and sets back farther than if level.

For the subject of backwater, Works on Hydraulics must be consulted.

Other applications of the level line are to obtain "contour lines" for topographical maps, for levees in irrigated rice-fields, etc.

To Run a Grade-line.—This consists in setting a series of pegs so that their tops shall be points in a line, which shall have any required slope ascending or descending.

First drive pegs at each end of a line to the heights required. These heights may differ by a given amount, or this difference may be undetermined.

Set the level over one of the pegs and measure the height, *a*, of the cross-wires above the top of the peg.

Set the rod on the other peg, and make the reading on the rod equal to the height *a*.

Without disturbing the level drive any desired number of pegs along the line, so that the reading on each will also be equal to a.

A line of uniform grade or slope is not a straight line.

Calling the globe spherical, this line when traced in the plane of a great circle would be a logarithmic spiral. On a length of six miles the distance of its middle point from the middle of its straight chord would be six feet almost exactly.

Correction for the Earth's Curvature and for Refraction.

This is necessary for long distances.

Let AB (Fig. 55) represent a portion of a section of the earth's surface. Then if a level be set at A, the line of sight of the level will be the tangent AD, while the true level will be the arc AB. The difference BD between the line of sight and the true level is the correction for the earth's curvature for the distance AB. This must be subtracted from the reading of the rod at B, or, what is the same thing, added to the height of B, as given by the reading of the rod.

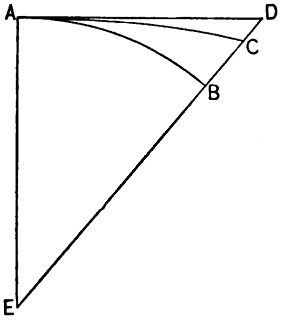


FIG 55.

Let $AE = R$, $AB = D$, and $BD = E$. By geometry,

$$\overline{AD}^2 = BD(BD + 2R),$$

or $$BD = \frac{\overline{AD}^2}{BD + 2R}.$$

Omitting BD in the right-hand member, since it is small compared with $2R$, and supposing $AD = AB = D$, we obtain

$$E = \frac{D^2}{2R} = \frac{D^2}{2 \times 20913650} = .000000023908D^2. \quad . \quad . \quad (1)$$

This formula gives a result or value for E slightly too small; but the relative error is only about one in 24,000 for a distance of $\frac{R}{100} = 39.6$ miles, or arc of $\frac{57^\circ.29578}{100} = 34'\ 22''.65.$

In observing distant objects, a ray of light traversing the air from an object to the eye or instrument is refracted, and takes a curved path which, for points near the surface of the earth, is practically the arc of a circle, concave downward, and whose radius is $7R$.

Thus a point at C (Fig. 55) would appear at D higher than it really is by an amount CD. This may be found from the above formula by substituting $7R$ for R.

Hence the correction for refraction is

$$E' = \frac{D^2}{14R} = .000000003415D^2. \quad . \; . \; . \; . \quad (2)$$

The correction for curvature and refraction is

$$E'' = BC = BD - CD = \frac{D^2}{2R} - \frac{D^2}{14R} = \frac{3D^2}{7R} = .000000020492D^2. \quad (3)$$

This must be added to the apparent elevation of the observed object to give the true elevation.

Table XI gives the value of the correction for the value of $R = 20911790$ feet.

When it is possible to set the level midway between the points whose heights are required, the corrections will balance each other and may be omitted.

The above equations may be put into a form sometimes more convenient as follows:

The length of arc on the earth's surface subtending angle of one minute is

$$\frac{2\pi \times R}{360 \times 60} = 6083 \text{ feet} = D_1, \text{ say.}$$

Then $$\frac{D_1^2}{14R} = .12638 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1')$$

= the correction for refraction for distance 6083 ft. or arc of 1′.

Also $$\frac{D_1^2}{2R} = .8846 \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2')$$

= the correction for curvature for the same distance or arc 1′, and $$\frac{3D_1^2}{7R} = .75828 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3')$$

= the correction for curvative and refraction.

TRIGONOMETRIC LEVELING.

First Method.—When the point C (Fig. 56) can be seen from two points A and B on the same level, then

$$AD = CD \cot CAD, \quad \text{and} \quad BD = CD \cot CBD.$$

Subtracting gives

$$AB = CD(\cot CAD - \cot CBD), \text{ or } CD = \frac{AB}{\cot CAD - \cot CBD}. \qquad (4)$$

Second Method.—Let A and B (Fig. 57) occupy any positions except in line with C. Measure AB and the angles at A and B; also the angle of elevation CAD.

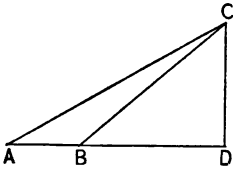


FIG. 56.

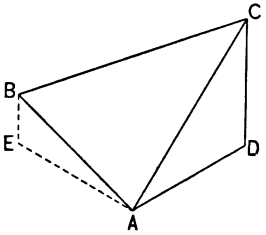


FIG. 57.

$$C = 180° - A - B, \quad AC = AB\frac{\sin B}{\sin C}, \text{ and } CD = AC \sin CAD. \qquad (5)$$

If A, B, and C are in the same vertical plane, the solution is in no wise affected.

Unless the distance $AD = D$ is short it is necessary to add to the correction found by the preceding formulas the correction for curvature and refraction, namely, $\frac{3D^2}{7R} = .000000020492D^2$, *To find the height of instrument by an observation of the horizon* (*Fig.* 58).

First Method.—Let C be the place of the transit, and BAA' a portion of a section of the earth's surface.

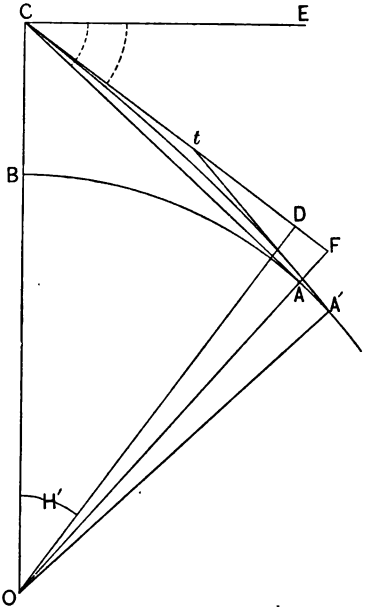


FIG. 58.

Were there no refraction the line of sight would be the tangent CA; $ACE = C$ would be the angle of depression or dip; and we would have

$$BC = BO \times \text{exsec } COA$$
$$= R \text{ exsec } C.$$

Owing to refraction, however, the line of sight would be a curve concave downward whose radius $= 7R$; and it would therefore extend from C to a point A', say a distance AA' beyond A.

Draw CDF tangent to this curve at C to meet the radius AO prolonged in F. Draw also the tangent $A't$.

Let $COA = H$, $COD = H'$, and $COA' = O$. Then

$$CO = r \sec H = \frac{r}{\cos H}.$$

Let E, on $A'O$ prolonged but not shown, be the center of the arc $A'C$. Now

$$\cos O = -\cos COE = \frac{\overline{CE}^2 - \overline{EO}^2 - \overline{CO}^2}{2CO \, . \, EO}$$

$$= \frac{13r^2 - r^2 \sec^2 H}{12r^2 \sec H} = \frac{13 \cos H - \sec H}{12}.$$

Clearing, substituting $1 - \text{vers } O$ for $\cos O$, $1 - \text{vers } H$ for $\cos H$, and $1 + \text{exsec } H$ for $\sec H$, we find

$$12 \text{ vers } O = 13 \text{ vers } H + \text{exsec } H.$$

Or, writing versines for exsecants or *vice versa*, we have

$$\text{vers } O = \tfrac{7}{6} \text{ vers } H,$$

or, approximately,

$$\text{exsec } O = \tfrac{7}{6} \text{ exsec } H.$$

In the triangle CEO,

$$\cos E = \frac{49r^2 + 36r^2 - \overline{CO}^2}{2 \times 6r \times 7r} = \frac{85r^2 - r^2 \sec^2 H}{84r^2} = \frac{85 - \sec^2 H}{84}.$$

Also $$\sin E = \sin O \frac{CO}{CE} = \sin O \frac{\sec H}{7}.$$

$$\therefore \sin E \sin O = \sin^2 O \frac{\sec H}{7} = (1 - \cos^2 O) \frac{\sec H}{7}$$

$$= \frac{170 - 169 \cos^2 H - \sec^2 H}{144} \times \frac{\sec H}{7}.$$

Now since OA' is the prolongation of EO, and EC and OD are perpendicular to CD and therefore parallel, we have

$$COA' - DOA' = COD, \quad \text{or} \quad O - E = H';$$

$$\therefore \cos H' = \cos (O - E) = \cos O \cos E + \sin O \sin E.$$

Substituting in this the above general values of $\cos O$, $\cos E$, and $\sin E \sin O$, and expanding and reducing, we find

$$\cos H' = \frac{13 \cos H + \sec H}{14} = \cos H + \frac{(\sec H - \cos H)}{14}.$$

Putting $\cos H = 1 - \text{vers } H$, $\sec H = 1 + \text{exsec } H$, etc., we find

$$\text{vers } H' = \tfrac{6}{7} \text{ vers } H - \frac{(\text{exsec } H - \text{vers } H)}{14}, \quad . \quad . \quad (a)$$

or $$\text{vers } H' = \tfrac{6}{7} \text{ vers } H, \text{ very nearly}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or $$\text{exsec } H' = \tfrac{6}{7} \text{ exsec } H, \text{ very nearly}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence $$\text{vers } H' = \tfrac{6}{7} \text{ vers } H - \frac{\text{exsec } H' - \text{vers } H'}{12}.$$

From this we have

$$\text{vers } H = \tfrac{7}{6} \text{ vers } H' + \frac{\tfrac{7}{6}(\text{exsec } H' - \text{vers } H')}{12}, \quad . \quad (b)$$

or $$\operatorname{vers} H = \tfrac{7}{6} \operatorname{vers} H', \text{ very nearly}, \quad . \quad . \quad . \quad . \quad . \qquad (6')$$

or $$\operatorname{exsec} H = \tfrac{7}{6} \operatorname{exsec} H', \text{ very nearly}. \quad . \quad . \quad . \quad . \quad . \qquad (7')$$

Since the versines of small angles are very nearly in the ratio of the squares of the magnitudes of the angles, we have, from (6),

$$H = \sqrt{\tfrac{7}{6}}H' = 1.08H', \text{ approximately}. \quad . \quad . \quad . \qquad (7')$$

Supposing $r = 4000$ miles and $AB = \dfrac{r}{80} = 50$ miles, then it is easy to show that the error of eq. (7) is less than .00000002, and the error of (6) is about .0000000004.

Example 1.—The observed dip of the sea horizon is $H' = 24'$. What is the height of the instrument above the sea?

We have

$$BC = r \operatorname{exsec} H = r\tfrac{7}{6} \operatorname{exsec} H' = 20914000 \times .00028467$$
$$= 595.35 \text{ feet, exactly}.$$

Example 2.—Let $r = 4000$ miles, and $AB = 50$ miles. What is the observed angle of depression H', and what is the height of the observer above the sea?

We have

$$H = \frac{50}{4000} \times 57°.29578 = 0°.7162 = 42' \, 58''.$$

$$\therefore \; H' = 0.716 \div 1.08 = 0°.663 \qquad = 39' \, 47''.$$

Also,

$$h = r \operatorname{exsec} H = 4000 \times .0000781 = .3124 \text{ miles} = 1649.47 \text{ feet}.$$

The exact relations between H and H', shown above, would seem to be more satisfactory than the approximate equations in general use even if these were regarded as sufficiently accurate.

THE STADIA.

The stadia is a compound cross-wire ring or diaphragm having three horizontal wires.

The two outer ones are called stadia wires, and distances determined by means of them are called stadia measurements.

The stadia wires are adjusted so as to intercept a certain space on a rod at a given distance from the transit and perpendicular to the line of sight.

Let C (Fig. 59) = the distance of the object-glass from the axis of the transit, and f = the focal length of the object-glass.

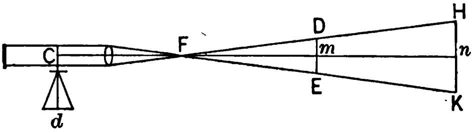


FIG. 59.

This focal length is equal to the distance of the cross-wires from the object-glass when this is focused for a distant object.

This focal length may be found also by removing the object-glass, exposing it to the rays of the sun, and noting at what distance from the center of the glass the rays form a perfect and minute image of the sun on a smooth surface.

Let $Cm = l'$, $Cn = l$, $DE = S'$, and $HK = S$.

The focal distance OF is constant, but $Co = c$ varies with the position of the object-glass, and hence CF is also variable.

Let $Co = c'$ when the rod is at DE, and $Co = c$ when the rod is at HK.

Now, from the figure,

$$\frac{Fn}{Fm} = \frac{HK}{DE}, \quad \text{or} \quad \frac{l-(c+f)}{l'-(c'+f)} = \frac{S}{S'},$$

or

$$l-(c+f) = \frac{S}{S'}[l'-(c'+f)]. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

S' is usually assumed = 1 foot, and $Fm = l' - (c' + f) = 100$ feet; and the stadia wires are then adjusted accordingly.

c' is measured on the telescope when the object-glass is focused on the rod at the assumed distance.

To measure any other distance the rod is again observed at the desired point and the space S noted, which placed in (8) gives $l - (c + f) = l_1$, say. We may then measure c on the telescope. Then

$$l = c + f + l_1.$$

Since, however, c has but a small range of values, it will usually be sufficient to assume it to be constant and equal to some mean value.

Suppose that in (8) $c = c' = c_1$, and solving we find

$$c_1 + f = \frac{Sl' - S'l}{S - S'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we observe S' and S corresponding to any two distances l' and l and substitute in (9), we have $c_1 + f$.

Having found $c_1 + f$, lay off $Cm = 100 + c_1 + f$, or $Fm = 100$, and adjust the stadia wires to subtend $DE =$ just one foot at that distance.

Then from (8), writing c_1 for c and c', we have

$$l - (c_1 + f) = l' - (c_1 + f)\frac{S}{S'} = 100S, \text{ or } l = 100S + (c_1 + f),$$

or, omitting accent,

$$l = 100S + (c + f). \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Example.—Suppose at $l' = 100$ we find $S' = 1$, and at $l = 500$ we find $S = 5.0453$.

Then eq. (9) gives

$$c + f = \frac{504.53 - 500}{4.0453} = 1.12.$$

Then, from eq. (10), $l = 100S + 1.12$, provided the stadia wires are spaced so as to intercept 1 foot at 101.12 feet distance from the center of the instrument.

The foregoing formulas must be modified when the line of collimation is oblique to the horizon, which is usually the case. Thus, in Fig. 60, let $D'E'$ = the space intercepted on the rod when the line of collimation FH is horizontal, and let $DE = S =$ the space intercepted when the line of collimation Fn makes an angle $nFH = a$ with the horizon.

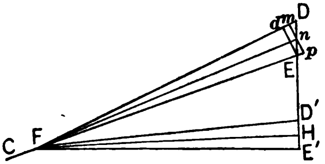


FIG. 60.

Let $DFE = D'FE' = \theta$.

In Fig. 60,

$$S = DH - EH = HF\,[\tan(a + \tfrac{1}{2}\theta) - \tan(a - \tfrac{1}{2}\theta)].$$

The horizontal reading desired is $D'E' = 2HF \tan \tfrac{1}{2}\theta$.

Dividing gives $$\frac{D'E'}{S} = \frac{2 \tan \frac{1}{2}\theta}{\tan(a + \frac{1}{2}\theta) - \tan(a - \frac{1}{2}\theta)}.$$

But $$2 \tan \tfrac{1}{2}\theta = \frac{2 \sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta}.$$

Also, by Chapter III, formulas (26), (27),

$$\tan(a + \tfrac{1}{2}\theta) - \tan(a - \tfrac{1}{2}\theta) = \frac{\sin \theta}{\cos(a + \frac{1}{2}\theta)\cos(a - \frac{1}{2}\theta)}$$

$$= \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{\cos^2 a - \sin^2 \frac{1}{2}\theta}.$$

Substituting these values, we obtain

$$\frac{D'E'}{S} = \frac{\cos^2 a - \sin^2 \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

If we neglect $\sin^2 \tfrac{1}{2}\theta$, or, what is equivalent, add $\sin^2 \tfrac{1}{2}\theta$ to the numerator, we introduce a relative error of

$$\frac{\sin^2 \frac{1}{2}\theta}{\cos^2 a - \sin^2 \frac{1}{2}\theta} = \frac{\sin^2 \frac{1}{2}\theta}{\cos^2 a} = \sin^2 \tfrac{1}{2}\theta \sec^2 a, \text{ very nearly.}$$

Again, if we add $\sin^2 \tfrac{1}{2}\theta$ to the denominator, making it unity, we introduce a relative error of $\dfrac{\sin^2 \frac{1}{2}\theta}{\cos^2 \frac{1}{2}\theta} = \sin^2 \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta$.

The first of these errors increases the fraction, and the second decreases it. Moreover, since $\theta = \dfrac{57^\circ.29578}{100} = 34'\ 22''.65$, we have in all practical cases $a > \tfrac{1}{2}\theta$, or $\sec^2 a > \sec^2 \tfrac{1}{2}\theta$.

Hence the fraction is, by the double approximation, increased *slightly* more than it is decreased. Hence

$$\frac{D'E'}{S} < \cos^2 a\,; \text{ but } \frac{D'E'}{S} = \cos^2 a, \text{ almost exactly.} \quad . \quad . \qquad (12)$$

The total relative error is

$$e = \sin^2 \tfrac{1}{2}\theta\,(\sec^2 a - \sec^2 \tfrac{1}{2}\theta)$$

$$= \sin^2 \tfrac{1}{2}\theta\,(\tan^2 a - \tan^2 \tfrac{1}{2}\theta),$$

Since $\tan^2 \frac{1}{2}\theta$ is very small compared with $\tan^2 a$, we have

$$e = \sin^2 \tfrac{1}{2}\theta \tan^2 a, \text{ very nearly,}$$

$$= .000012 \tan^2 a, \text{ very nearly.}$$

Hence

$$\frac{D'E'}{S} = \cos^2 a(1 - .000012 \tan^2 a)$$

$$= \cos^2 a - .000012 \sin^2 a. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Since $\sin \frac{1}{2}\theta = \tan \frac{1}{2}\theta = \dfrac{1}{200} = .005$, very nearly, the quantity neglected above is only $(.005)^4 = .00000000625$ and does not come within the range of the table.

The above is the coefficient of reduction by which to multiply the observed space $DE = S$ in order to produce the true space $D'E'$ which would be observed at the same distance if the line of collimation were horizontal.

Hence we have, from (10),

$$l = (100S + c + f)(\cos^2 a - .000012 \sin^2 a).$$

By eqs. (25) and (26), Chap. III,

$$\cos^2 a = \frac{1 + \cos 2a}{2} = 1 - \frac{\text{vers } 2a}{2},$$

and

$$\sin^2 a = \frac{1 - \cos 2a}{2} = \frac{\text{vers } 2a}{2}.$$

Hence

$$l = (100S + c + f)\left(1 - \frac{\text{vers } 2a}{2} - .000012\,\frac{\text{vers } 2a}{2}\right), \quad . \quad (14)$$

or

$$l = (100S + c + f)\left(1 - \frac{\text{vers } 2a}{2}\right), \text{ very nearly.} \quad . \quad . \quad . \quad (14')$$

These coefficients may be read from a table of versed sines without any computation whatever.

The last equation is quite accurate enough, but the coefficients of Table XIII are calculated by the exact formula.

Example.—Find the coefficient for $a = 7° 20'$.

We write half the vers 14° 40'.................... = .01629
and subtract from unity and find coefficient......... = .98371

Another method of procedure is that in which the rod is held perpendicular to the line of collimation.

To secure this position of the rod a bar is attached to it having sights upon it, through which the rodman watches the instrument during an observation, the line of sights being perpendicular to the rod.

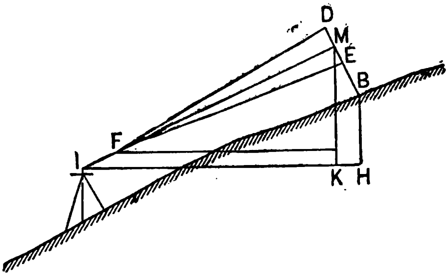


FIG. 61.

The horizontal distance of the point B from the instrument is

$$IH = IK + KH = Im \cos a + Bm \sin a,$$

or

$$IH = (100S + c + f) \cos a + r \sin a.$$

(See Fig. 61, in which r is the reading of the rod by the line of collimation.)

The elevation of B above I is

$$BH = mK - Bm \cos a,$$

or

$$BH = (100S + c + f) \sin a - r \cos a. \quad . \quad . \quad . \quad (15)$$

When the distances are sufficiently great correction, must be made for curvature and refraction as already pointed out.

THE GRADIENTER.

This attachment consists of a screw working against a clamping-arm suspended from the horizontal axis, on the opposite end from the vertical arc.

A strong spiral spring presses the arm against the end of the screw. The large silvered head of the screw is usually divided into 100 equal spaces. When the screw is turned the head moves along one division of a small silvered scale for each revolution of the screw.

When the regular clamp of the telescope is free the telescope may be revolved; but when the telescope is held by this clamp it can be moved only by the gradienter screw, which thus takes the place of the vertical tangent-screw.

The screw is cut with such a "pitch" as to cause the horizontal cross-wire to move over a space of 1 foot on a rod 100 feet distant when the screw is turned through one complete revolution. The wire will therefore move $\frac{1}{100}$ of a foot on the rod 100 feet distant when the screw is turned through $\frac{1}{100}$ of a revolution, which is equal to one of the small spaces on the head.

Grades may be established with great facility with this screw. First, set the screw at zero and level and clamp the telescope. Then move it upward or downward as many spaces of the screw as there are hundredths of a foot to the hundred feet of the required grade. Thus for a grade of 1.25 of a foot turn the screw through 125 spaces, or one revolution and one fourth of a revolution.

This screw is also very convenient for measuring short distances. Since one revolution of the screw will move the wire over one foot at the distance of 100 feet, it is only necessary to turn the screw through one revolution and observe the number of feet the wire moves on the rod at the required distance, and multiply that by 100.

Thus suppose the space on the rod is 2.57 feet, then the distance = 2.57 × 100 = 257 feet.

When the distance to be measured is not approximately horizontal some modifications are necessary, which we will now explain.

In Fig. 62 let O be the position of the transit; $DOB = E =$ the angle of elevation to the foot of the rod; $AOB = \theta =$ the angle subtended by any number of revolutions of the gradienter screw; $AB = S =$ the space on the rod subtended by the angle θ when the rod is vertical. BC is drawn perpendicular to BO. Then

$$ABC = BOD = E;$$
$$OAB = 90^\circ - (E + \theta);$$

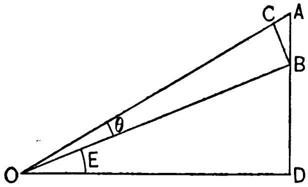


Fig. 62.

$$\frac{BC}{AB} = \frac{\sin\,[90^\circ - (E + \theta)]}{\sin\,(90^\circ + \theta)} = \frac{\cos E \cos \theta - \sin E \sin \theta}{\cos \theta}.$$

$$\therefore\ BC = S(\cos E - \sin E \tan \theta);$$

$$BO = \frac{BC}{\tan\theta} = S\left(\frac{\cos E}{\tan\theta} - \sin E\right);$$

$$DO = BO\cos E = S\left(\frac{\cos^2 E}{\tan\theta} - \tfrac{1}{2}\sin 2E\right).$$

These formulas are applicable to any gradienter screw.

Let $\tan\theta = \frac{1}{100}$ for one revolution of the screw. Then we have

$$BO = S(100\cos E - \sin E); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$DO = S(100\cos^2 E - \tfrac{1}{2}\sin 2E) = S\left(\frac{100(1+\cos 2E)}{2} - \tfrac{1}{2}\sin 2E\right). \quad (17)$$

This last equation is decidedly the simplest formula for computation, since it involves $2E$ only and no exponent.

When the angle θ is an angle of *depression* the point B is the upper end of the part AB of the rod used. (See Fig. 63.) The horizontal distance only is usually required; and Table XIV gives, for different values of E, the coefficient $\frac{100(1+\cos 2E)}{2} - \frac{1}{2}\sin 2E$ by which to multiply the readings of the rod in order to find the horizontal distances.

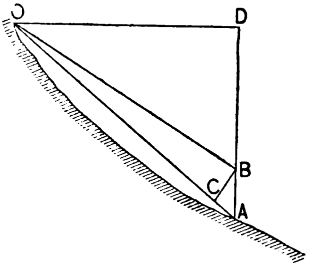


FIG. 63.

VERTICAL CURVES.

Vertical curves are used to round off the angles formed by the meeting of two grades.

Let AC and CB (Fig. 64) be two grades meeting at C. These grades are determined by the rise or fall per station in going in either direction. Thus, in the figure, let g represent the *rise* per station of the grade of AC, and g' represent the *fall* per station of the grade CB. In going in the opposite direction BC would, of course, be a rising grade and CA a falling grade.

The parabola furnishes a very suitable vertical curve, and it is very easily computed and laid out.

Let AEB represent the required parabola. Draw the vertical line CH, also a vertical line through B to meet AC prolonged in

F, and the horizontal lines AK and CG to meet BF in K and G. Join AB, and drop the perpendiculars $a_1m_1b_1$ and $a_2m_2b_2$. Let $n =$ the number of stations in AC or CB. Then, since these distances are measured horizontally, we have $AH = HK$, and therefore $AD = DB$.

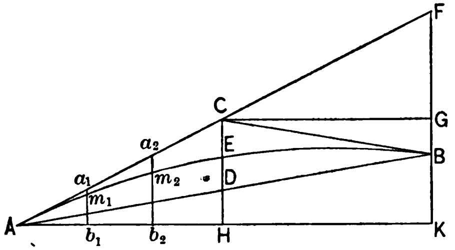


FIG. 64.

The vertical line CD is therefore a diameter of the parabola, and the distances of points on the curve in a vertical direction from corresponding points on the tangent AF are in the ratio of the squares of the distances of these points from A. Also,

$$CD = \tfrac{1}{2}FB; \quad CE = \tfrac{1}{4}FB = \tfrac{1}{2}CD;$$

$$\therefore \quad CE = ED.$$

Now, since $CF = CA$,

$FG = CH =$ the rise of AC in n stations $= ng$.

Also, $GB =$ the fall of GB in n stations $= ng'$;

$$\therefore \quad FB = n(g + g').$$

Now, since B is $2n$ stations from A, we have

Offset at first station from $A = a_1m_1 = a = \dfrac{FB}{4n^2} = \dfrac{g + g'}{4n}$. (18

The value of a_1m_1 being determined, the distances of the curve at all points from the tangent AF are also known. Thus

$$a_2m_2 = 4a, \quad Ce = 9a, \text{ etc.}$$

It is easy to calculate the heights of the points of the curve above AK, though these heights are scarcely needed. Thus

$$m_1b_1 = a_1b_1 - a_1m_1 = g - a;$$

$$m_2b_2 = a_2b_2 - a_2m_2 = 2g - 4a;$$

$$EH = CH - CE = 3g - 9a, \text{ etc., etc.}$$

Finally,

$$BK = FK - FB = 2ng - 4n^2a = 2ng - n(g + g') = n(g - g'),$$

or

$$BK = CH - GB = ng - ng' = n(g - g') \text{ as before.}$$

The successive grades are found by taking the successive differences of the heights just found. Thus

$$m_1b_1 - o = g - a\,;\ m_2b_2 - m_1b_1 = g - 3a\,;\ EH - m_2b_2 = g - 5a, \text{ etc.}$$

The change in the grades from station to station, which is the same as the chord deflection of the curve, is, we observe, equal to the constant quantity $2a$.

Second Proof.—The curve must change its direction in its length of $2n$ stations an amount equal to $g \pm g'$, and therefore in each station it must change an amount equal to $\frac{g \pm g'}{2n}$; and this is equal to $2a$ by eq. (18).

Third Proof.—The ordinates to the curve are $a, 4a, 9a, 16a$, etc. The differences of ordinates are $3a, 5a, 7a$, etc.

∴ the differences of grades are $2a$, $2a$, etc., a constant.

Fourth Proof.—Let m_2 represent the nth station, and prolong the chord m_1m_2 to meet CE in x (not shown).

Now $a_1m_1 = (n - 1)^2a$, $a_2m_2 = n^2a$, and $CE = (n + 1)^2a$.

But

$$Cx + a_1m_1 = 2a_2m_2, \quad \text{or} \quad Cx = 2a_2m_2 - a_1m_1 = [(n + 1)^2 - 2]a.$$

Hence $\qquad Ex = CE - Cx = 2a.$

In finding the value of a, etc., it is necessary to know when we are to take the sum of the grades, and when the difference; for there may be four combinations of two adjacent grades.

It is only necessary to observe:

(1) That when the grades are both ascending or descending FB is equal to $n(g - g')$, g being the steeper of the two grades, and therefore

$$a = \frac{n(g - g')}{4n^2} = \frac{g - g'}{4n}.$$

(2) That when one of the grades is ascending and the other descending,

$$FB = n(g + g'), \text{ and } \quad a = \frac{g + g'}{4n}.$$

If one of the tangents is horizontal, let g = grade of the other and then $a = \frac{g}{4n}$.

In all cases it is necessary to consider only the *change* of grade at C; which we will represent by G. Then

$$G = g - g', \quad \text{or} \quad G = g + g',$$

according as the grades are both rising or falling, or one rising and the other falling. Then

$$FB = nG, \text{ and } \quad a = \frac{FB}{4n^2} = \frac{G}{4n}.$$

These formulas apply without change to points between stations, as the following examples will illustrate.

Example 1.—A .9% up grade joins a .3% down grade at station 76 at an elevation of 94.0. Find ordinates, etc., for a curve 6 stations long.

$$a = \frac{.9 + .3}{12} = .1. \quad \text{(See Fig. 64.)}$$

Stations on AF........	73	74	75	76	77	78	79
Elevations on AF.....	91.3	92.2	93.1	94.0	94.9	95.8	96.7
Corrections............	0 0	.1	.4	.9	1.6	2.5	3.6
Elevations on curve....	91.3	92.1	92.7	93.1	93.3	93.3	93.1

It is necessary to compute the ordinates a, $4a$, etc., for one half the curve only, since they are the same for corresponding points on each tangent. Thus we have:

Stations on AC and CB	73	74	75	76	77	78	79
Elevations on AC and CB	91.3	92.2	93.1	94.0	93.7	93.4	93.1
Corrections............	0.0	.1	.4	0.9	.4	.1	0.0
Elevations on curve....	91.3	92.1	92.7	93.1	93.3	93.3	93.1

Example 2.—A .2% up grade is continued beyond station 17, whose elevation is 46.4, by an .8% up grade. Find ordinates, etc., for a curve 6 stations long.

$$a = \frac{.8 - .2}{12} = .05.$$

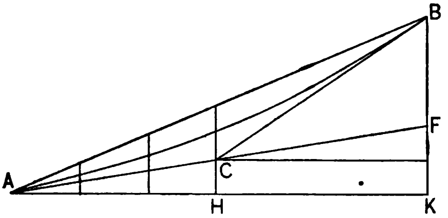


FIG. 65.

Stations on AC and CB..	14	15	16	17	18	19	20
Elevations on AC and CB	45.8	46.0	46.2	46.4	47.2	48.0	48.8
Ordinates	0.0	.05	.20	.45	.20	.05	0.0
Elevations on curve.....	45.8	46.05	46.4	46.85	47.4	48.05	48.8

To find the grade of the parabola at any point.

Let the curve be that of Example 1, and let the point be at $a_1 + 40$, or 140 feet from A.

Elevation at $a_1 + 40 = 92.2 + .40 \times .9 = 92.2 + .36 = 92.56$.

Also, $(1.4)^2 a = 1.96a = .196 = .20$, nearly.

Therefore elevation of curve at $a_1 + 40 = 92.36$.

The special need of these curves is in sags in the grades to prevent the breaking of the train.

According to Wellington, *Railway Location*, page 365, vertical curves in sags should be *at least* 200 feet long, or 100 feet on each side of the vertex, for each tenth in the rate of change of grade.

This would call for a curve 1200 feet long in the last example.

It is not always practicable to meet this requirement, but such curves could be and should be much longer than they are generally made.

ELEVATION OF THE OUTER RAIL ON CURVES.

A car of weight w moving on a curve of radius R with a velocity of v feet per second develops a centrifugal force in the direction ab (Fig. 66) expressed by

$$f = \frac{wv^2}{32.15R}.$$

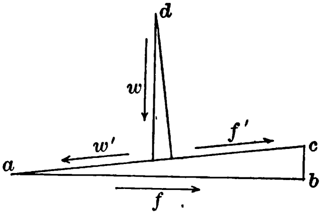


FIG. 66.

To counteract this force the outer rail on a curve is raised above the inner rail an amount $bc = e$, so that the car may rest on an inclined plane.

Let $ac = g$, the gauge of the track.

The component of f in the direction of ac is

$$f\frac{ab}{ac} = f\frac{ab}{g} = f' = \frac{wv^2}{32.15R} \cdot \frac{ab}{g}.$$

The component of w in the direction of ca is

$$w' = w\frac{bc}{ac} = w\frac{e}{g}.$$

Now f' and w' are opposite in direction, and in order to satisfy the mechanical conditions they must be equal.

Equating their values, therefore, we find

$$e = \frac{abv^2}{32.15R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

But ac = distance between rail centers = gauge + one rail head = 4.708 + .202 = 4.91. For an elevation of 6 inches

$$ab = ac - .03 = 4.88, \text{ nearly.}$$

This is a good average value for ab.

Again, if V = the velocity in miles per hour, we have

$$v = \frac{5280}{3600}V = \frac{22}{15}V.$$

Furthermore, $R = \frac{R_1}{D}$, in which D is the degree of a curve of radius R, and R_1 = the radius of a one-degree curve = 5729.58. Substituting these values in (19), we have

$$e = \frac{4.88 \times 484 D V^2}{32.15 \times 225 \times 5729.58} = .00005699 D V^2$$
$$= .000057 D V^2, \text{ almost exactly. } \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If we substitute in (19) $ab = g$, and $v = \frac{22}{15} V$, we have, approximately,

$$e = \frac{.06688 g V^2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is the formula in general use.

To give the most favorable view possible of this formula, substitute in it $g = 4.708$ and find

$$e = \frac{.3149 V^2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Example.—Find the elevation, e, for a 7° curve and a velocity of 40 miles per hour.

Eq. (20) gives

$$e = .000057 \times 7 \times 1600 = .638 \text{ feet} = 7.66 \text{ inches.}$$

Eq. (22) gives

$$e = \frac{.3149 \times 1600}{818.5} = .616 \text{ feet} = 7.39 \text{ inches.}$$

It will be observed that the two large factors appearing in (22) do not occur in (20); and, moreover, eq. (20) is very much more accurate than the formula (22) generally used.

The author has elsewhere pointed out the evil of elevating the outer rail on sharp curves sufficiently to meet the requirements of too high velocities.

The elevation should be much less than required for the speed of the fastest passenger-coaches; for it is better, on such a curve, for a coach to hug the outer rail somewhat or to slacken speed, or both, than to pull all the freight-cars against the inner rail.

The best conditions are realized when the speeds of fast trains are lessened so as to exceed only slightly the speed for which the elevation of the curve is suited.

Thus a train moving at a speed of 50 miles per hour should be decreased to a 35-mile rate or less in passing over a curve elevated for a 30-mile rate. Under such conditions the cars would slightly press the outer rail around the curve; their motion would thus be steadied, and the movement would be the steadiest possible.

The maximum elevation should probably not exceed eight inches, except, possibly, at some special point, under peculiar conditions.

CHAPTER VI.

COMPOUND CURVES.

A COMPOUND curve consists of two or more consecutive circular arcs of different radii; any two adjacent ones having a common tangent at their point of meeting, and their centers on the same side of that tangent.

This common point of tangency is called a point of compound curve, or P. C. C.

Compound curves are used to bring the line of the road upon more favorable ground than any simple curve could occupy.

Let *AP* and *PB* (Fig. 67) be the two branches of a compound curve uniting the two tangents *AV* and *BV*. Continue *AP* to *k*, where the tangent *kD* is parallel to *BV*, and draw the chords *AP*, *Pk*, and *PB*.

To prove that the chords *Pk* and *PB* coincide.

Since the radii PO_1 and PO_2 are each perpendicular to the common tangent at *P*, they coincide; and since the radii O_1k and O_2B are perpendicular to parallel tangents *Dk* and *BV*, they are parallel. Hence the vertex angles of the isosceles triangles PO_1k and PO_2B are equal, and the angles at the bases must be equal also. Hence $O_1Pk = O_2PB$, and therefore *Pk* and *PB* coincide.

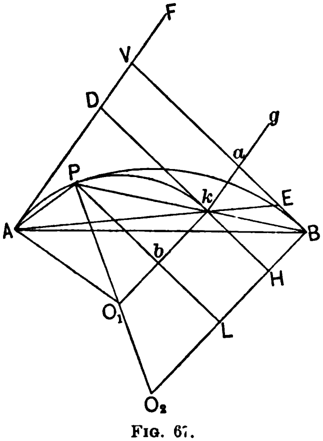


FIG. 67.

Prolong *Ak* to meet *BV* in *E*. Let R_1 = the first radius AO_1, and R_2 = the second radius PO_1O_2. Let $O_1 = AO_1P$, the central angle, subtended by the arc of radius R_1, and $O_2 = PO_2B$, the central angle subtended by the arc of radius R_2.

Now $AkP (= BkE) = \frac{1}{2}O_1$, by (*e*), Chapter IV.

Also, $kAP = PkD = PBV = \frac{1}{2}O_2$, by (*d*), Chapter IV.

In the triangle ABV let the tangent $AV = T_1$, and the tangent $BV = T_2$. We observe that the shorter tangent is adjacent to the shorter radius, and the longer tangent is adjacent to the longer radius. Let $AB = c$, $BAV = A$, $ABV = B$.

$$\therefore\ BVF = V = A + B = O_1 + O_2.$$

Draw kag parallel to AV.

Now $$\frac{aB + ak}{aB - ak} = \frac{\tan\frac{1}{2}(akB + aBk)}{\tan\frac{1}{2}(akB - aBk)} = \frac{\cot\frac{1}{2}(akB - aBk)}{\cot\frac{1}{2}(akB + aBk)}.$$

See eq. (35), Chapter III.

$$ak + aB = AV + BV - (AD + Dk) = T_2 + T_1 - 2AD$$
$$= T_2 + T_1 - \frac{2R_1}{\cot\frac{1}{2}V};$$

$$aB - ak = BV - AV = T_2 - T_1.$$

Also

$$aBk = DkP = \tfrac{1}{2}O_2, \quad \text{and} \quad akB = V - \tfrac{1}{2}O_2.$$

Hence

$$akB - aBk = V - O_2 = O_1, \quad \text{and} \quad akB + aBk = V.$$

Substituting these values in the above equation gives

$$\frac{T_2 + T_1}{T_2 - T_1} - \frac{2R_1}{(T_2 - T_1)\cot\frac{1}{2}V} = \frac{\cot\frac{1}{2}O_1}{\cot\frac{1}{2}V}. \quad . \quad . \quad (a)$$

Hence

$$\cot\tfrac{1}{2}O_1 = \frac{(T_2 + T_1)\cot\frac{1}{2}V - 2R_1}{T_2 - T_1},$$

or

$$\left.\begin{aligned} R_1 &= \tfrac{1}{2}(T_2 + T_1)\cot\tfrac{1}{2}V - \tfrac{1}{2}(T_2 - T_1)\cot\tfrac{1}{2}O_1 \\ &= T_1\cot\tfrac{1}{2}V - \tfrac{1}{2}(T_2 - T_1)(\cot\tfrac{1}{2}O_1 - \cot\tfrac{1}{2}V). \end{aligned}\right\} \quad . \quad . \quad (1)$$

Similarly,

$$\cot \tfrac{1}{2}O_2 = \frac{(T_2 + T_1)\cot \tfrac{1}{2}V - 2R_2}{T_1 - T_2},$$

or

$$\left.\begin{aligned} R_2 &= \tfrac{1}{2}(T_2 + T_1)\cot \tfrac{1}{2}V - \tfrac{1}{2}(T_1 - T_2)\cot \tfrac{1}{2}O_2 \\ &= T_2 \cot \tfrac{1}{2}V - \tfrac{1}{2}(T_1 - T_2)(\cot \tfrac{1}{2}O_2 - \cot \tfrac{1}{2}V). \end{aligned}\right\} \quad . \; . \; . \; . \quad (2)$$

From (1) and (2) we get

$$\left.\begin{aligned} (R_2 - R_1) &= \tfrac{1}{2}(T_2 - T_1)(\cot \tfrac{1}{2}O + \cot \tfrac{1}{2}O_2), \\ \text{or} \qquad \frac{R_2 - R_1}{T_2 - T_1} &= \frac{\cot \tfrac{1}{2}O_1 + \cot \tfrac{1}{2}O_2}{2}. \end{aligned}\right\} \quad . \; . \; . \quad (3)$$

We observe that either radius can be found by two multiplications and both by three multiplications, since $\tfrac{1}{2}(T_2 + T_1)\cot \tfrac{1}{2}V$ is common. This is one multiplication less than is necessary by any other formulas. The first equations of groups (1) and (2) are preferable when both radii are to be found, but the last equation of either group is best when only one radius is required. Since

$$\tfrac{1}{2}V = \tfrac{1}{2}O_1 + \tfrac{1}{2}O_2 < 90^\circ, \quad \tfrac{1}{2}O_1 < 90^\circ, \quad \text{and} \quad \tfrac{1}{2}O_2 < 90^\circ.$$

Supposing the triangle ABV unchanged; then, since $O_1 + O_2$ is constant, O_2 varies inversely with O_1.

Eq. (1) shows that R_1 varies directly with O_1, and eq. (2) shows that R_2 varies inversely with O_2 and therefore directly with O_1. Hence the radii vary directly with the central angle corresponding to the shorter radius. This shows the advantage of making O_1 as large as possible.

Prolong the tangent Dk to meet BO_2 in H, and draw the perpendicular PbL. Now

$$BH = BL - kb = R_2 \text{ vers } O_2 - R_1 \text{ vers } O_2 = (R_2 - R_1) \text{ vers } O_2.$$

Then $$DV = \frac{BH}{\sin V} = \frac{(R_2 - R_1) \text{ vers } O_2}{\sin V},$$

and $$AD = R_1 \tan \tfrac{1}{2}V = \frac{R_1 \text{ vers } V}{\sin V}.$$

Hence

$$\left.\begin{aligned}(AD+DV)\sin V &= T_1 \sin V\\ &= R_1(\operatorname{vers} V - \operatorname{vers} O_2) + R_2 \operatorname{vers} O_2.\\ \text{Then}\quad \operatorname{vers} O_2 &= \frac{T_1 \sin V - R_1 \operatorname{vers} V}{R_2 - R_1},\\ \text{and}\quad R_2 &= \frac{T_1 \sin V - R_1(\operatorname{vers} V - \operatorname{vers} O_2)}{\operatorname{vers} O_2}.\end{aligned}\right\} \quad (4)$$

Similarly

$$\left.\begin{aligned}T_2 \sin V &= R_1 \operatorname{vers} O_1 + R_2(\operatorname{vers} V - \operatorname{vers} O_1),\\ \text{or}\quad R_1 &= \frac{T_2 \sin V - R_2(\operatorname{vers} V - \operatorname{vers} O_1)}{\operatorname{vers} O_1},\\ \text{or}\quad \operatorname{vers} O_1 &= \frac{T_2 \sin V - R_2 \operatorname{vers} V}{R_1 - R_2}.\end{aligned}\right\} \quad (5)$$

It should be noticed that the foregoing equations are symmetrical with reference to the radii, the tangents, and the central angles, and hence eq. (2) may be written at once from eq. (1) or *vice versa*, by simply exchanging the radii, the tangents, and the central angles; and the same is true of eqs. (4) and (5).

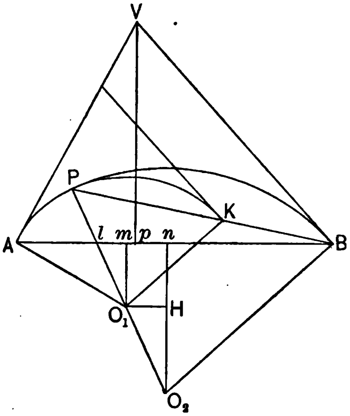


FIG. 68.

Let Fig. 68 represent a compound curve, in which $AO_1 = R_1$, $BO_2 = R_2$, $P =$ the P.C.C.

Drop the perpendiculars O_1m and O_2n upon the chord AB, and draw O_1H parallel to AB. Let l be the intersection of AB and PO_1, and l the angle AlP. We have

$$O_2H = O_1O_2 \sin l = (R_2 - R_1)\sin l.$$

Also, $$O_2H = O_2n - O_1m = R_2 \cos B - R_1 \cos A.$$

Equating these gives

$$\sin l = \frac{R_2 \cos B - R_1 \cos A}{R_2 - R_1}.$$

Again, $Am = R_1 \sin A,$

$$mn = O_1H = O_1O_2 \cos l = (R_2 - R_1) \cos l,$$

$$nB = R_2 \sin B.$$

Adding gives

$$C = R_1 \sin A + R_2 \sin B + (R_2 - R_1) \cos l \quad . \quad . \quad . \quad (6)$$

Now $\quad T_1 = C \frac{\sin B}{\sin V}, \quad \text{and} \quad T_2 = C \frac{\sin A}{\sin V}.$

When R_1, R_2, A, and B are given, l, C, and the tangents T_1 and T_2 may be found from the above equations.

We have

$$l = O_1Al + AO_1l = (90^\circ - A) + O_1, \quad \text{or} \quad O_1 = A + l - 90^\circ.$$

Also,

$$l + lBO_2 + lO_2B = 180^\circ, \quad \text{or} \quad l + (90^\circ - B) + O_2 = 180^\circ,$$

or $$O_2 = B - l + 90^\circ.$$

When R_1, R_2, A, and B are given, we may find l, O_1, and O_2 from these equations, and then T_1 and T_2 from (4) and (5).

To find the limit in one direction of each radius of a compound curve.

We observe that a curve run from A (Fig. 67) with a radius $R' = T_1 \cot \frac{1}{2} V$ would be tangent to BV at E, VE being equal to VA.

A curve run from A or from any other *point* of *this curve* with a radius greater than $T_1 \cot \frac{1}{2} V$ would lie outside of the above curve and would intersect VE. Hence the smaller radius must be less than $T_1 \cot \frac{1}{2} V$.

Again, a curve run from B with a radius $R'' = T_2 \cot \frac{1}{2} V$ would be tangent to VA prolonged at B' (VB' being equal to VB) and would pass inside of A.

A curve run from B or from any other point of *this curve* with a radius less than $T_2 \cot \frac{1}{2} V$ would lie inside of the above curve and would pass still further from A. Hence the greater radius must be greater than $T_2 \cot \frac{1}{2} V$.

From trigonometry we have

$$\cot (45^\circ - z) + \cot (45^\circ + z) = 2 \sec 2z > 2,$$

Therefore

$$\cot (45° - a - z) + \cot [(45 - a + z)] > 2 \sec 2z > 2.$$

Hence if the sum of two angles $< 90°$, the sum of their cotangents > 2, and the more unequal are the angles the greater is the sum of their cotangents.

Now since $V < 180°$, $\frac{1}{2}O_1 + \frac{1}{2}O_2 = \frac{1}{2}V < 90°$.

Therefore $\cot \frac{1}{2}O_1 + \cot \frac{1}{2}O_2 > 2$ in all cases, and eq. (3) shows that

$$R_2 - R_1 > T_2 - T_1 \text{ in all cases. . . . . } \quad (e)$$

Again from eqs. (1) and (2),

$$R_2 > T_2 \cot \tfrac{1}{2}V, \quad \text{and} \quad R_1 < T_1 \cot \tfrac{1}{2}V.$$

Therefore $$\frac{R_2}{R_1} > \frac{T_2 \cot \frac{1}{2}V}{T_1 \cot \frac{1}{2}V} = \frac{T_2}{T_1}. \quad . \ . \ . \ . \ . \ . \ . \quad (f)$$

These equations show that the radii differ more than the tangents do, both relatively and absolutely.

Putting $O_1 = O_2$ in (3) gives $\dfrac{R_2 - R_1}{T_2 - T_1} = \cot \frac{1}{2}O_1 = \cot \frac{1}{2}V$.

The following table shows the value of this ratio for some values of V.

V	$\dfrac{R_2 - R_1}{T_2 - T_1}$
120°	1.73
100	2.14
80	2.75
60	3.73
40	5.67
30	7.60
20	11.43
15	15.26

For the more common values of V, say 40° or less, we see that the difference between the radii is several times greater than the difference between the tangents; and we may remember that when O_1 and O_2 are unequal the ratios of $\dfrac{R_2 - R_1}{T_2 - T_1}$ are somewhat larger than those given in the table. But the tabular results are sufficiently accurate, and they will aid much in assigning practical values to the radii when the tangents are known.

Of the quantities R_1, R_2, O_1, and O_2, and the sides and angles of the triangle ABV, four independent quantities must be given in order to find the others. Or three besides V must be given. Of course the parts of the triangle must be consistent, and all of the given quantities must be consistent with the necessary relations existing among them.

For example, we must have $A + B = O_1 + O_2 = V$,

V is supposed to be known in all cases. If either O_1 or O_2 is known, the other is known from the relation above.

PROBLEMS.

I. Given T_1 and T_2, also either one of the quantities R_1, R_2, O_1, and O_2, to find the others.

Example 1.—$T_1 = 447.32$, $T_2 = 510.84$, $V = 15°$, and $R_1 = 3000$.

$$\therefore\ T_2 + T_1 = 958.1,\quad T_2 - T_1 = 63.52,\ \tfrac{1}{2}(T_2 - T_1) = 31.76,$$

and
$$\cot \tfrac{1}{2}V = 7.596.$$

(1) gives

$$\cot \tfrac{1}{2}O_1 = \frac{958.1 \times 7.596 - 6000}{63.52} = \frac{7277.64 - 6000}{63.52} = 20.115,$$

and
$$\tfrac{1}{2}O_1 = 2° \, 50' \, 45''.$$

Now $\tfrac{1}{2}O_2 = \tfrac{1}{2}V - \tfrac{1}{2}O_1 = 4° \, 39' \, 15''$, and $\cot \tfrac{1}{2}O_2 = 12.283$.

$$\therefore\ \cot \tfrac{1}{2}O_1 + \cot \tfrac{1}{2}O_2 = 32.398.$$

From (3), $R_2 = 31.76 \times 32.398 + 3000 = 4028.96$.

If R_2 is given, find in the same way $\tfrac{1}{2}O_2$, $\tfrac{1}{2}O_1$, and R_1.
If O_1 is given, $O_2 = V - O_1$.
Then find R_1 from (1), and R_2 from (2).
Similarly if O_2 is given.

Example 2 (see Henck, Prob. 48).—Let $T_1 = 480$, $T_2 = 500$, $V = 18°$, and $O_1 = \tfrac{1}{2}V = 9°$.

$$\therefore\ O_2 = 9°.$$

In this case the common tangent at the P.C.C. makes equal angles with the tangents AV and BV. We have

$$\tfrac{1}{2}(T_2 + T_1) = 490,\quad \tfrac{1}{2}(T_2 - T_1) = 10,$$

$$\cot \tfrac{1}{2}V = 6.31375,\quad \cot \tfrac{1}{2}O_1 = \cot \tfrac{1}{2}O_2 = 12.7062.$$

(1) gives

$$R_1 = 490 \times 6.31375 - 10 \times 12.7062 = 2966.67.$$

(3) gives

$$R_2 = 10(12.7062 + 12.7062) + 2966.67 = 3220.79.$$

II. Given A, B, and C and any one of the quantities R_1, R_2, O_1, and O_2, to find the others.

Example 1.—Let $A = 8°$, $B = 7°$, and $C = 950$. Let $R_1 = 3000$ We have

$$T_1 = C\frac{\sin B}{\sin V} = \frac{950}{.25882} \times .12187 = 3670.5 \times .12187 = 447.32;$$

$$T_2 = \frac{C \sin A}{\sin V} = 3670.5 \times 13917 = 510.84.$$

Eq. (1) gives

$$\cot \tfrac{1}{2}O_1 = \frac{(T_2 + T_1)\cot \tfrac{1}{2}V - 2R_1}{T_2 - T_1}$$

$$= \frac{958.16 \times 7.59575 - 6000}{63.52} = \frac{1277.94}{63.52} = 20.1188.$$

$\therefore \tfrac{1}{2}O_1 = 2° \, 50' \, 45''$; $\therefore \tfrac{1}{2}O_2 = 4° \, 39' \, 15''$, and $\cot \tfrac{1}{2}O_2 = 12.2836$.

Eq. (3) gives

$$R_2 = \frac{T_2 - T_1}{2}(\cot \tfrac{1}{2}O_1 + \cot \tfrac{1}{2}O_2) + R_1$$
$$= 31.76 \times 32.4024 + 3000 = 4028.96.$$

If R_2 is given, find T_1 and T_2 as above, and then O_2, O_1, and R_1 in order.

If O_1 is given, we have $O_2 = (A + B) - O_1$.

Find T_1 and T_2 as shown, then R_1 and R_2 from (1) and (2). Similarly if O_2 is given.

Example 2.—Let $O_1 = A = 8°$; then $O_2 = B = 7°$, and $C = 950$. Let $R = 3000$.

In this case the common tangent at the P.C.C. is parallel to the chord AB. T_1 and T_2 are found as above. Then eq. (1) gives

$$R_1 = \tfrac{1}{2}(T_2 + T_1)\cot \tfrac{1}{2}V - \tfrac{1}{2}(T_2 - T_1)\cot \tfrac{1}{2}O_1$$
$$= 479.08 \times 7.59575 - 31.76 \times 14.3007$$
$$= 3638.97 - 454.19 = 3184 \, 8,$$

and

$$R_2 = \frac{T_2 - T_1}{2}(\cot \tfrac{1}{2}O_1 + \cot \tfrac{1}{2}O_2) + R_1$$
$$= 31.76(14.3007 + 16.35) + 3184.8 = 4158.26.$$

One of the most useful problems in compound curves is that of fitting such a curve to the ground. The lengths of the branches give the corresponding central angles, the sum of which is equal to the deflection angle V.

We select a problem proposed by "L. E. Vel" in *Engineering News*, March 5, 1881.

III. "Two tangents AV and BV intersect at an angle of 60°. It is required to begin at the point A and locate 2000 feet of 2° curve, followed by 2000 feet of 1° curve.

"Required the distance to bo measured back from the intersection V to the point of curve A by the *shortest* and *simplest* method."

A solution of this problem, marvelous for length and labor involved, can be seen in *Engineering News* of March 19, 1881. (See also Searle's *Field Engineering*, Art. 164.)

Being challenged by the contriver of the above solution, the author gave, in *Engineering News* of April 2, 1881, in substance the following as the "shortest and simplest" solution possible:

We have $R_2 = 2R_1$, or $R_2 - R_1 = R_1$.

Substituting R_1 for $R_2 - R_1$ in (4), observing that $O_2 = 20°$, we have

$$\begin{aligned} T_1 &= R_1 (\text{vers } 20° + \text{vers } 60°) \div \sin 60° \\ &= 2864.8 \times .56031 \div 86603 = 1853.5. \qquad \text{Q.E.D.} \end{aligned}$$

If, for any reason, T_2 is desired, we have, from (5),

$$T_2 = \frac{2R_1 \text{ vers } V - R_1 \text{ vers } O_1}{\sin V} = \frac{R_1(1 - \text{vers } O_1)}{\sin V} = \frac{R_1 \cos O_1}{\sin V}$$

$$= 2864.8 \times .76604 \div .86603 = 2534.04.$$

IV. Given R_1 and R_2 and either tangent, to find the other tangent and the central angles O_1 and O_2.

Example.—Let $V = 72°$, $T_1 = 1091$, $R_1 = 954.9$, and $R_2 = 3437.7$. $\sin V = \sin 72° = .95106$, $\text{vers } V = .69098$, and $R_2 - R_1 = 2482.8$.

(4) gives

$$\text{vers } O_2 = \frac{1091 \times .95106 - 954.9 \times .69098}{2482.8} = .15216.$$

$$\therefore \quad O_2 = 32° \, 1' \, 20'' \quad \text{and} \quad O_1 = V - O_2 = 39° \, 58' \, 40'';$$

$$\text{vers } O_1 = .23371.$$

Now (5) gives

$$T_2 = \frac{954.9 \times .23371 + 3437.7 \times .45727}{.95106} = 1887.5.$$

V. Given either tangent and the adjacent or the opposite radius, also either central angle O_1 or O_2, to find the other tangent and radius.

Example 1.—Given $AV = T_1 = 1091$, $R_1 = 954.9$, $V = 72°$, and $O_1 = 39° \, 58' \, 20''$. Hence $O_2 = 32° \, 1' \, 40''$.

(4) gives

$$R_2 = \frac{1091 \times .95106 - 954\ 9(.69098 - .15216)}{.15216} = 3437.7.$$

Then T_2 is found from (5) as above.

Example 2.—Given $T_1 = 1091$, $R_2 = 3437.7$, and the angles as above.

From (4),

$$R_1 = \frac{1091 \times .95106 - 3437.7 \times .15216}{.53882} = 954.9.$$

Then T_2 is found from (5) as above.

VI. Given the tangents and a central angle O_1 or O_2, to find the radii.

R_1 and R_2 are found from (1) and (2).

Example.—Let $T_1 = 480$, $T_2 = 500$, $V = 18°$ and $O_1 = 9°$.

Then $$O_2 = 18° - 9° = 9°.$$

Now $\cot \frac{1}{2}V = 6.31375$; $\cot \frac{1}{2}O_1 = \cot \frac{1}{2}O_2 = 12.7062$.

Now, from (1),

$$R_1 = 240 \times 19.02 - 250 \times 6.39245 = 2966.7;$$

and from (2),

$$R_2 = 250 \times 19.02 - 240 \times 6.39245 = 3220.8.$$

SPECIAL PROBLEMS.

It frequently happens that a curve already located must be changed to meet certain conditions. In this case the equation must be solved anew or the results already obtained must be modified.

If the required changes are small, and especially if some latitude is allowed in making them, the solution can sometimes be readily modified to conform to the new conditions; but it is generally best to solve the equation anew.

This will be illustrated in the following problems.

Having located a simple curve APB, radius $AO_1 = R_1$, ending in the tangent BV, it is required to end with a given shorter radius, $B'O' = R_2$, in a tangent $B'V'$, parallel to BV, and at a distance *within* BV equal to $BH = p$.

The distance BP back to the P.C.C., and the distance $HB' = d$, that the end of the new curve is back of the old, are required.

Let $PO'B' = PO_1B = O$.

Now

$$\cos O = \frac{KO_1}{O_1O'} = \frac{BO_1 - HK - BH}{O_1O'}$$

$$= \frac{R_1 - R_2 - BH}{R_1 - R_2} = 1 - \frac{p}{R_1 - R_2},$$

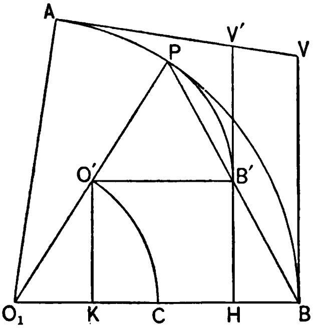


FIG. 69.

or $$\operatorname{vers} O = \frac{p}{R_1 - R_2}, \quad \text{or} \quad R_1 - R_2 = \frac{p}{\operatorname{vers} O},$$

or $$R_2 = R_1 - \frac{p}{\operatorname{vers} O}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Or proceed as follows: Describe the arc $O'C$ with center O_1. Now

$$\operatorname{vers} O = \frac{CK}{O_1O'}.$$

But $\quad CK = BK - BC = BK - PO' = BK - R_2,$

and $\quad BH = BK - HK = BK - R_2.$

$$\therefore \ CK = BH = p.$$

Hence, as before,

$$\operatorname{vers} O = \frac{p}{R_1 - R_2}.$$

Also, $\quad d = HB' = KO' = (R_1 - R_2) \sin O. \quad . \quad . \quad (8)$

From (7) and (8),

$$\frac{d}{p} = \frac{\sin O}{\text{vers } O} = \cot \tfrac{1}{2}O,$$

or $$d = p \cot \tfrac{1}{2}O. \qquad (9)$$

Equation (9) is easily proved as follows: Draw $PB'B$, which we have seen is a straight line. We have

$$BB'H = B'BV = \tfrac{1}{2}PO_1B = \tfrac{1}{2}O.$$

Hence $$HB' = d = HB \cot BB'H = p \cot \tfrac{1}{2}O.$$

Example 1.—Having located a 3° curve APB (Fig. 69), radius 1909.9, required the distance back from B to the beginning of a 5° curve (radius 1145.9) that will end in a tangent 27 feet within the tangent BV, also the distance HB'.

(7) gives

$$\text{vers } (PO'B' = PO_1B) = \tfrac{27}{764} = .03534;$$

$$\therefore\ O = 15^\circ\ 17'.$$

Then $$\frac{15^\circ\ 17'}{3^\circ} = 5.09 \text{ chains} = 509 \text{ feet} = \text{arc } BP.$$

Also, $$HB' = p \cot \tfrac{1}{2}O = 27 \times 7.453 = 201.2 \text{ feet.}$$

II. Suppose the curve APB is located, and it is desired to end with a longer radius, $PO' = R_2$, on a parallel tangent $B'V'$, at a distance $HB = p$ *without* BV.

Similar equations to the above apply.

Let $PO_1B = PO'B' = O$. Then, as in Fig. 69,

$$p = BH = B'K - BD$$
$$= R_2 \text{ vers } O - R_1 \text{ vers } O$$
$$= (R_2 - R_1) \text{ vers } O. \qquad (7')$$

Also, $$d = B'H = KD$$
$$= (R_2 - R_1) \sin O, \qquad (8')$$

and $$d = p \cot \tfrac{1}{2}O. \qquad (9')$$

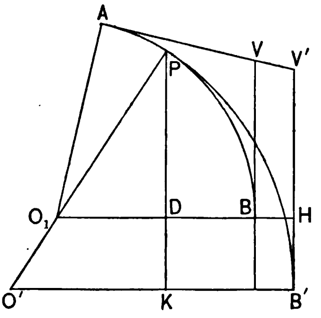


FIG. 70.

Example 2.—Having located a 5° curve APB, required the distance back from B to the beginning of a 3° curve that will end in a tangent 27 feet without BV.

$$PO_1B = PO'B' = O.$$

We have

$$\text{vers } O = \tfrac{27}{764} = .03534.$$

$$\therefore\ O = 15^\circ\ 17'.$$

Then $$\frac{15^\circ\ 17'}{5^\circ} = 3.06 \text{ chains} = 306 \text{ feet} = \text{arc } BP.$$

Also, $$B'H = p \cot \tfrac{1}{2}O = 27 \times 7.453 = 201.2 \text{ feet},$$

the distance that the new tangent point B' is in advance of the original tangent point B.

With reference to the preceding problems, we observe that if any two of the following quantities: the new radius, the offset p, the tangent distance $HB' = d$, the arc BP (or $B'P$) or the equivalent, the angle O, are given, the other two may be readily found by the use of the preceding equations.

Example 3.—Let $p = 1$, and let $PB = 150$ feet, the degree of the curve APB (Fig. 69) being 7° 10′. Hence

$$O = 10^\circ\ 45',\quad R_1 = 800.$$

Eq. (7) gives $$R_2 = 800 - \frac{1}{.01755} = 800 - 57 = 743.$$

Also, $$HB' = \cot 5^\circ\ 22\tfrac{1}{2}' = 10.63.$$

Example 4.—Having run two given curves, PB and PB' (Fig. 69), subtending equal central angles, $PO_1B = PO'B' = O$, to find the distance between the terminal tangents BV and $B'V'$ and the distance HB'.

(7) gives

$$p = (R_1 - R_2) \text{ vers } O.$$

Then, from (8), we have

$$HB' = (R_1 - R_2) \sin O.$$

These equations and Figs. 69 and 70 show that if a curve is sharpened or flattened, so as to end on a tangent parallel to the original tangent, the new curve does not end at a point opposite to the original, but is moved backward or forward according as the curve is sharpened or flattened.

Problem.—Given a compound curve APB' (Fig. 71) ending in the tangent $B'V'$, the first radius, $AO_1 = PO_1 = R_1$, being shorter than the second radius, $PO_2 = R_2$; to change the P.C.C. and also the last radius, so that the new curve radius $OP' = R'$ shall end in a given parallel tangent $B''V''$.

It is evident that the new P.C.C. must be on the first branch AP, since if on the second branch the problem would be the same as the preceding. The P.C.C. (at P') must be found so that the arc $P'B''$ will be tangent to $B''V''$.

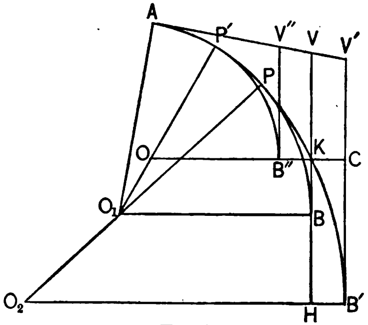


FIG. 71.

Suppose the first branch AP of the curve extended to B, where tangent BV is parallel to $B'V'$.

Let $AO_1P = O_1$, $PO_2B' = O_2$, and $P'OB'' = O$.

Let $B'H = a$, $B''K = a'$, $B''C = p$, $BH = d$, $BK = d'$.

I. Suppose $R' < R_1 < R_2$. Then $B''V''$ is necessarily *within* $B'V'$.

From the preceding problem we have directly

$$B'H = (R_2 - R_1) \text{ vers } O_2 = a; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$BH = O_1O_2 \sin O_2 = (R_2 - R_1) \sin O_2 = d. \quad . \quad . \quad (11)$$

Dividing gives

$$\frac{\text{vers } O_2}{\sin O_2} = \tan \tfrac{1}{2}O_2 = \frac{a}{d}, \quad \text{or} \quad d = a \cot \tfrac{1}{2}O_2. \quad . \quad . \quad (12)$$

a and d are expressed in terms of known quantities, and are always known.

Again,

$$B''K = (R_1 - R') \text{ vers } O = a', \quad \text{or} \quad \text{vers } O = \frac{a'}{R_1 - R'}; \quad . \quad (13)$$

$$BK = OO_1 \sin O = (R_1 - R') \sin O = d' \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\therefore \tan \tfrac{1}{2}O = \frac{a'}{d'}, \quad \text{or} \quad d' = a' \cot \tfrac{1}{2}O. \quad . \quad . \quad . \quad (15)$$

Since the position of the tangent is given, a' is known and eq. (13) determines the beginning of the curve. Also,

$$HK = d + d' = a \cot \tfrac{1}{2}O_2 + a' \cot \tfrac{1}{2}O, \quad . \quad . \quad (16)$$

which determines the end of the curve.

2. If the curve is to begin at a certain point, O is known, and we have, from the above eq. (13),

$$\text{radius} = R' = R_1 - \frac{a'}{\text{vers } O}.$$

Then the end of the curve is found as above.

3. If the curve is to end at a given distance back (or front) of B', d' is known, and we have

$$\tan \tfrac{1}{2}O = \frac{a'}{d'},$$

which determines the beginning of the curve.

Then the radius, R', is given as above.

II. Suppose $R' > R_1$. (Fig. 72.)

(a) Let $B''V''$ be within $B'V'$. Then $R' < R_2$.

We have
$$B'H = (R_2 - R_1) \text{ vers } O_2 = a; \quad . \quad . \quad . \quad . \quad (10')$$
$$BH = (R_2 - R_1) \sin O_2 = d. \quad . \quad . \quad . \quad . \quad (11')$$
$$\therefore \tan \tfrac{1}{2}O_2 = \frac{a}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12')$$

Also, $$B''K = (R' - R_1) \text{ vers } O = a'; \quad . \quad . \quad . \quad . \quad (13')$$

$$BK = (R' - R_1) \sin O = d'; \quad . \quad . \quad . \quad . \quad (14')$$

and $$\tan \tfrac{1}{2}O = \frac{a'}{d'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15')$$

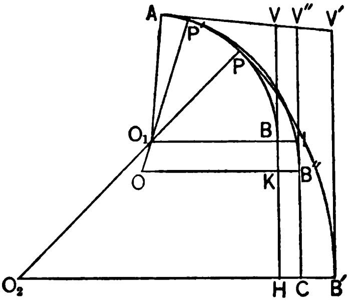


FIG. 72.

Also, $\text{vers } O = \dfrac{a'}{R' - R_1}$, or $R' = R_1 + \dfrac{a'}{\text{vers } O}$.

Finally, $$HK = BH - BK = d - d'. \quad . \quad . \quad . \quad . \quad (16')$$

2. If the beginning of the curve is fixed, O is known and the radius

$$= R' = R_1 + \frac{a'}{\text{vers } O}.$$

$$HK = d - d' = a \cot \tfrac{1}{2}O_2 - a \cot \tfrac{1}{2}O,$$

giving end of the curve.

3. If the end of the curve is fixed, d' is known and

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}.$$

Then $$R' = R_1 + \frac{a'}{\text{vers } O}.$$

(*b*) Suppose $B''V''$ is without $B'V'$. (Fig. 73.)

In this case we may have

$$R' \overset{<}{\underset{>}{=}} R_2.$$

We have, as already explained,

$$B'H = (R_2 - R_1) \text{ vers } O_2 = a, \quad . \quad . \quad . \quad (10'')$$

$$BH = (R_2 - R_1) \sin O_2 = d, \quad . \quad . \quad . \quad . \quad (11'')$$

and

$$\tan \tfrac{1}{2} O_2 = \frac{a}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12'')$$

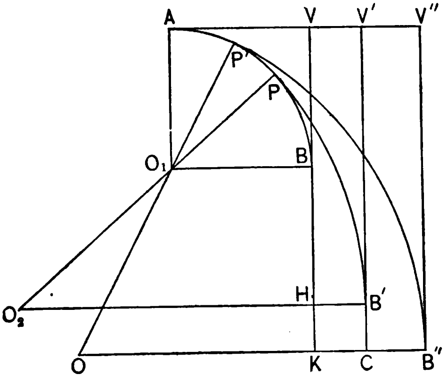


FIG. 73.

Also,

$$B''K = (R' - R_1) \text{ vers } O = a', \quad . \quad . \quad . \quad (13'')$$

and

$$BK = (R' - R_1) \sin O = d', \quad . \quad . \quad . \quad . \quad (14'')$$

and

$$\tan \tfrac{1}{2} O = \frac{a'}{d'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15'')$$

By transposition,

$$\text{vers } O = \frac{a'}{R' - R_1},$$

which determines the beginning of the curve.

By subtraction,

$$HK = d' - d, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16'')$$

which determines the end of the curve.

A special case of this problem is that in which the P. C. C. only is changed. Then $R' = R_2$, as illustrated in Fig. 73.

$B''V''$ is without $B'V'$. (Fig. 73.)

(*a*) Substituting R_2 for R' in the value of a', we have

$$a' - a = p = (R_2 - R_1)(\text{vers } O - \text{vers } O_2).$$

$$\therefore \quad \text{vers } O = \text{vers } O_2 + \frac{p}{R_2 - R_1}.$$

Then $$R_2 = R_1 + \frac{a'}{\text{vers } O}.$$

$P'O_1P = O - O_2$ shows the angular distance that P' is back of P.

(*b*) We may consider that the curve $AP'B''$ is first located, and that APB' is the new curve, the new tangent $B'V'$ being *within* $B''V''$. Then

$$P'OB'' = O_2, \quad \text{and} \quad PO_2B' = O,$$

and we have

$$\text{vers } O = \text{vers } O_2 - \frac{p}{R_2 - R_1}.$$

$$R_2 = R_1 + \frac{a'}{\text{vers } O}, \text{ as before.}$$

$P'O_1P$ shows the angular distance that the new P.C.C. (P) is in advance of P'.

2. If, in this problem, the beginning of the curve is fixed, O is known and we have

$$R' = R_1 + \frac{a'}{\text{vers } O}.$$

The end of the curve is given by

$$HK = d' - d = a' \cot \tfrac{1}{2}O - a \cot \tfrac{1}{2}O_2.$$

3. If the end of the curve is fixed, d' is known and

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}.$$

This determines the beginning of the curve.

Then the radius $= R' = R_1 + \dfrac{a'}{\text{vers } O}$.

4. A special case of the general problem is that in which the new curve ends on the same radial line as the original curve.

(*a*) OB'' coincides with O_2B', and H with K, as shown in the figure. Hence

$BK = BH$, or $d' = d$.

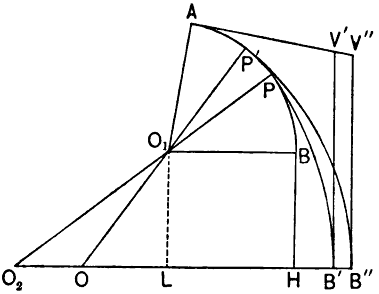


FIG. 74.

We have from equations preceding, by subtracting and transposing,

$$\tan \tfrac{1}{2}O = \tan \tfrac{1}{2}O_2 + \frac{a'}{d'} - \frac{a}{d} = \tan \tfrac{1}{2}O_2 + \frac{a' - a}{d} = \tan \tfrac{1}{2}O_2 + \frac{p}{d}.$$

This gives the beginning of the curve. We have

$$O_1L = O_1O_2 \sin O_1O_2L = O_1O \sin O_1OL,$$

or $$(R_2 - R_1) \sin O_2 = (R' - R_1) \sin O.$$

$$\therefore\ R' = \frac{(R_2 - R_1) \sin O_2}{\sin O} + R_1.$$

$P'O_1P = O - O_2$ shows the angular distance that P' is back of P.

(*b*) We may suppose the curve $AP'B''$ first located, and that APB' is the new curve, the new tangent $B'V'$ being within $B''V''$. Then

$$P'OB'' = O_2, \quad PO_2B' = O.$$

$$\therefore\ \tan \tfrac{1}{2}O = \tan \tfrac{1}{2}O_2 - \frac{p}{d},$$

and
$$R' = \frac{(R_2 - R_1)\sin O_2}{\sin O} + R_1,$$

as before found.

In this case the new P. C. C. (P) is on the arc AP'' prolonged.

From the above value of R_1' we have

$$\frac{\sin O}{\sin O_2} = \frac{R_2 - R_1}{R' - R_1},$$

which shows that for O and O_2, each less than 90°, we have

$$O \overset{>}{\underset{<}{=}} O_2 \text{ accordingly as } R_2 \overset{>}{\underset{<}{=}} R'.$$

Hence the radius and the P. C. C. must both be changed.

EXAMPLES.

1. A 10° 4′ curve, AP, is followed by 100 feet of 7° 44′ curve, PB'. It is required to find the beginning, the end, and the length of a 17° 48′ curve that will end in a parallel tangent, $B''V''$, 58 feet inside of $B'V'$. (Fig. 71.)

We have

$$R_1 = 569.2, \qquad R_2 = 740.9, \quad \text{and} \quad R' = 321.9;$$

also, $O_2 = 7°\,44'$, and $p = 58$ feet.

Hence, eq. (10),

$$B'H = (R_2 - R_1) \text{ vers } O_2 = 171.7 \times .00909 = 1.56 = a,$$

and, eq. (13),

$$B''K = 58 - 1.56 = 56.44 = a',$$

and therefore

$$\text{vers } O = \frac{56.44}{247.3} = .22822. \quad \therefore \quad O = 39°\,29'.$$

Now

$$39°\,29' - 7°\,44' = 31°\,45', \text{ and } 100\frac{31°\,45'}{10°\;\;4'} = 315.4 \text{ ft.}$$

Hence the beginning of the new curve is 315.4 ft. back of P.

Also, eqs. (11) and (14),

$$HK = (R_2 - R_1) \sin O_2 + (R_1 - R') \sin O$$

$$= 171.7 \times .1346 + 247.3 \times .6358 = 180.3 +,$$

or $$HK = 1.56 \times 14.79 + 56.44 \times 2.788 = 180.4.$$

The length of $P'B''$ is $100\frac{39° \ 29'}{17° \ 48'} = 221.8$ feet.

2. Suppose the original curve and offsets a and a' as above, and that the new curve is to begin at P', 315.4 feet back of P. To find R' and the end of the curve. We have

$$P'O_1P = 10° \ 04' \times 3.154 = 31° \ 45'.$$

$$\therefore \ P'OB'' = 7° \ 44' + 31° \ 45' = 39° \ 29' = O.$$

Then, eq. (13),

$$R' = R_1 - \frac{a'}{\operatorname{vers} O} = 569.2 - \frac{56.44}{.22822} = 321.9.$$

The distance HK is now found as above.

3. Suppose that it is required to end the curve a given distance, HK, back of B'. Then

$$d' = HK - d; \quad \tan \tfrac{1}{2}O = \frac{a'}{d'}; \quad \text{and} \quad R' = R_1 - \frac{a'}{\operatorname{vers} O}.$$

Problem.—Given a compound curve, APB', ending in the tangent $B'V'$, the first radius, $AO_1 = R_1$, being longer than the second radius, $PO_2 - R_2$, to change the P. C. C., and also the last radius, so that the new curve shall end in a given parallel tangent $B''V''$.

I. Suppose $R' > R_1 > R_2$. Then $B''V''$ is without $B'V'$. Let the notation be as in the preceding problems. Then

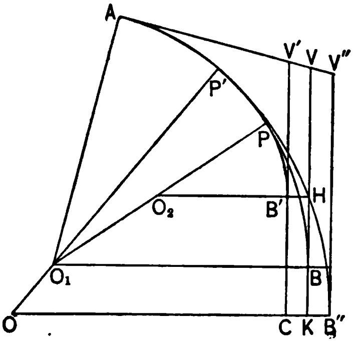


FIG. 75.

$$B'H = (R_1 - R_2) \text{ vers } O_2 = a; \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$BH = O_1O_2 \sin O_2 = (R_1 - R_2) \sin O_2 = d. \quad . \quad (11a)$$

a and d are expressed in terms of known quantities and are always known.

Dividing gives

$$\frac{\text{vers } O_2}{\sin O_2} = \tan \tfrac{1}{2}O_2 = \frac{a}{d} \quad \text{or} \quad d = a \tan \tfrac{1}{2}O_2. \quad . \quad (12a)$$

Similarly,

$$B''K = (R' - R_1) \text{ vers } O = a', \quad \text{or} \quad \text{vers } O = \frac{a'}{R' - R_1}. \quad (13a)$$

$$BK = OO_1 \sin O = (R' - R_1) \sin O = d', \quad . \quad . \quad . \quad . \quad (14a)$$

and $$\tan \tfrac{1}{2}O = \frac{a'}{d'}, \quad \text{or} \quad d' = a' \text{ cotan } \tfrac{1}{2}O_1. \quad . \quad . \quad (15a)$$

Since the position of the tangent is given, a' is known, and then vers O determines the position of P' or the beginning of the new curve.

Again,

$$HK = d + d' = a \cot \tfrac{1}{2}O_2 + a' \cot \tfrac{1}{2}O. \quad . \quad . \quad (16a)$$

This determines the end of the curve.

2. If the beginning of the curve is assigned, O is known and we have

$$R' = R_1 + \frac{a'}{\text{vers } O}.$$

The end of the curve is given as above.

3. If the curve is to end at a given distance in front of B', d' is known and we have

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}.$$

This determines the beginning of the curve.

Then the radius $= R' = R_1 + \dfrac{a'}{\text{vers } O}$.

II. Suppose $R' < R_1$.

(*a*) Let the terminal tangent $B''V''$ be without $B'V'$, then $R' > R_2$.

Now

$$B'H = (R_1 - R_2) \text{ vers } O_2 = a; \quad (10b)$$

$$BH = O_1O_2 \sin O_2$$

$$= (R_1 - R_2) \sin O_2 = d; \quad (11b)$$

$$\tan \tfrac{1}{2}O_2 = \frac{a}{d},$$

or $$d = a \cot \tfrac{1}{2}O_2 . . \quad (12b)$$

Again,

$$B''K = (R_1 - R') \text{ vers } O = a',$$

or $$\text{vers } O = \frac{a'}{R_1 - R'} \quad (13b)$$

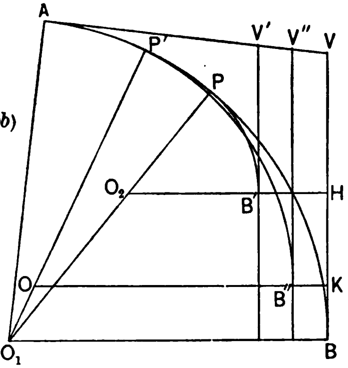


FIG. 76.

and $$BK = OO_1 \sin O = (R_1 - R') \sin O = d'; \; . \; . \; . \quad (14b)$$

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}, \quad \text{or} \quad d' = a' \cot \tfrac{1}{2}O. \; . \; . \; . \quad (15b)$$

$$\text{vers } O = \frac{a'}{R_1 - R'}$$

determines the beginning of the curve, and

$$HK = BH - BK = d - d' = a \cot \tfrac{1}{2}O_2 - a' \cot \tfrac{1}{2}O \quad (16b)$$

determines the end of the curve.

2. If the beginning of the curve is given, find $R' = R_1 - \frac{a'}{\text{vers } O}$, as before.

The end of the curve is given by $HK = d - d'$, as above.

3. If the end of the curve is given, d' is known, and then

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}.$$

This determines the beginning of the curve.

Then the radius is given as above.

(*b*) Suppose $B''V''$ to be within $B'V'$. We may have

$$R' \overset{<}{\underset{>}{=}} R_2.$$

Now $$B'H = (R_1 - R_2) \text{ vers } O_2 = a; \quad . \quad . \quad . \quad . \quad (10c)$$

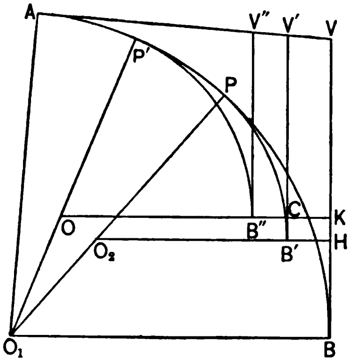


FIG. 77.

$$BH = O_1O_2 \sin O_2$$

$$= (R_1 - R_2) \sin O_2 = d; \quad (11c)$$

$$\tan \tfrac{1}{2}O_2 = \frac{a}{d},$$

or $$d = a \cot \tfrac{1}{2}O_2; \quad (12c)$$

$$B''K = (R_1 - R') \text{ vers } O = a',$$

or $$\text{vers } O = \frac{a'}{R_1 - R'}. \quad (13c)$$

$$BK = OO_1 \sin O$$

$$= (R_1 - R') \sin O = d'; \quad (14c)$$

$$\tan \tfrac{1}{2}O = \frac{a'}{d'},$$

or $$d' = a' \cot \tfrac{1}{2}O. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15c)$$

The position of the tangent gives a', and then eq. (13*c*) determines the beginning of the new curve.

$$HK = d' - d = a' \cot \tfrac{1}{2}O - a \cot \tfrac{1}{2}O_2 \quad . \quad . \quad (16c)$$

determines the end of the curve.

1. A special case of the above is that in which $R' = R_2$. Substituting R_2 for R' in the values of a' and a, and subtracting, we have

$$CB'' = p = a' - a = (R_1 - R_2)(\text{vers } O - \text{vers } O_2).$$

$$\therefore \quad \text{vers } O = \text{vers } O_2 + \frac{p}{R_1 - R_2},$$

and $$R' = R_2 = R_1 - \frac{a'}{\text{vers } O}.$$

We may regard $AP'B''$ as the original curve, APB' as the new curve, the new tangent $B'V'$ being without $B''V''$. Then

$$P'OB'' = O_2, \qquad PO_2B' = O;$$

and
$$\text{vers } O = \text{vers } O_2 - \frac{p}{R_1 - R_2},$$

$$R_2 = R_1 - \frac{a'}{\text{vers } O},$$

as before.

2. If in this problem the beginning of the curve is fixed, O is known and

$$R' = R_1 - \frac{a'}{\text{vers } O}.$$

The end of the curve is given as above.

3. If the end of the curve is fixed, d' is known and

$$\tan \tfrac{1}{2}O = \frac{a'}{d'}.$$

Then the radius is found as above.

4. A special case of the last is that in which the new curve ends on the same radial line as the original curve.

(*a*) OB'' coincides with O_2B', and H with K. Hence $d' = d$. Then

$$\tan O = \frac{a'}{d'} = \frac{a'}{d}; \tan O_2 = \frac{a}{d}.$$

$$\therefore \quad \tan O - \tan O_2 = \frac{a' - a}{d},$$

or

$$\tan \tfrac{1}{2}O = \tan \tfrac{1}{2}O_2 + \frac{a' - a}{d}$$

$$= \tan \tfrac{1}{2}O_2 + \frac{p}{d}.$$

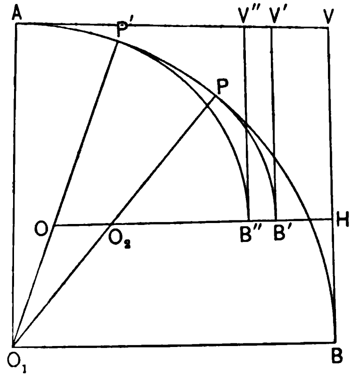


FIG. 78.

This gives the beginning of the curve. Now

$$BH = O_1O_2 \sin O_2 = O_1O \sin O = (R_1 - R_2) \sin O_2 = (R_1 - R') \sin O.$$

From this we have

$$R' = R_1 - \frac{(R_1 - R_2)\sin O_2}{\sin O}.$$

(*b*) We may regard $AP'B''$ as the original curve, and APB' as the new curve, the tangent $B'V'$ being without $B''V''$. Then

$$P'OB'' = O_2, \quad PO_2B' = O.$$

$$\therefore \quad \tan \tfrac{1}{2}O = \tan \tfrac{1}{2}O_2 - \frac{p}{d},$$

and $$R' = R_1 - \frac{(R_1 - R_2)\sin O_2}{\sin O}.$$

In laying out compound curves it is generally best to lay out each branch from the station at the beginning of the branch, and subsequent stations if necessary.

Thus, in Fig. 67, AP would be laid out from A, and other stations on AP if necessary. Then the transit would be moved to P, directed along the tangent at that point, and then deflected to stations on PB, etc.

We will show further on, however, how to lay out one branch of a compound curve from a station on another branch.

Problem.—Given a compound curve APB, ending in a tangent BV, to change the curve so as to end on a new tangent BV', but at the same tangent point as before.

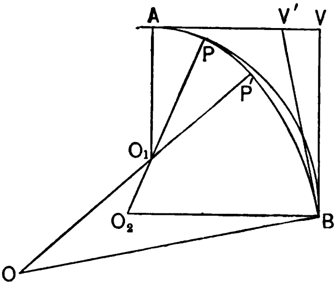


FIG. 79.

I. Suppose the shorter radius to precede the longer. (Fig. 79.)

Let B = the known angle VBV', $AV = T_1$, $BV = T_2$.

Let $AV' = T'$, $BV' = T''$ and $VV' = a$.

We have, from the figure,

$$V' = V - B.$$

In the triangle BVV' we have

$$\frac{T''}{T_2} = \frac{\sin V}{\sin V'}, \quad \text{or} \quad T'' = T_2\frac{\sin V}{\sin V'}.$$

Also, $$\frac{a}{T_2} = \frac{\sin B}{\sin V'}, \quad \text{or} \quad a = T_2\frac{\sin B}{\sin V'}.$$

Then $$T' = T_1 - a.$$

With these values of V', T', and T'' the tangents may be connected by assuming a value for one of the quantities O_1, O_2, R_1, or R_2, and using eq. (1) or (2) and (3).

When V' is on the prolongation of AV, we have

$$V' = V + B, \quad \text{and} \quad T' = T_1 + a.$$

Otherwise the solution is just the same as the above.

II. Suppose the longer radius to precede the shorter. (Fig. 80.)

We have

$AV = T_1$, $BV = T_2$, $AV' = T'$,

and $BV' = T''$.

Also, $VV' = a$,

and, in the figure, the angle $V' = V + B$.

Then $$T'' = T_2\frac{\sin V}{\sin V'};$$

$$a = T_2\frac{\sin B}{\sin V'}.$$

Also, $$T' = T_1 + a.$$

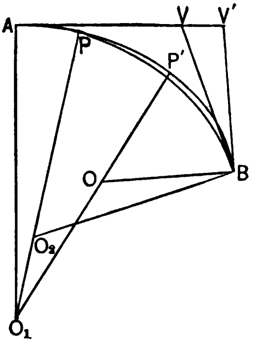


FIG. 80.

With these values of V', T', and T'' a compound curve may be located, as already explained.

Remark.—A special solution of this problem in a voluminous work on Field Engineering, with an example worked out, occupies six pages.

Problem.—Having found the radius $AO = R$ of a curve, to substitute for it two radii, $AO_1 = R_1$, and $P_1O_2 = R_2$, the longer of which, AO_1, is to be used for a certain distance only, at each end of the curve.

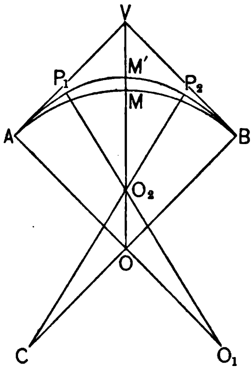


FIG. 81.

Let $AO_1P_1 = BCP_2 = O_1$,

and $P_1O_2P_2 = 2O_2$.

Supposing the arc AP_1 described with the longer radius, the intersection of P_1O_1 with the central radius MO of the simple curve in O_2 gives O_2 as the center, and P_1O_2 as the radius, of the central branch of the compound curve. In the triangle OO_1O_2,

$$O_1OO_2 = 180° - \tfrac{1}{2}V, \quad OO_1O_2 = O_1,$$

and $$OO_2O_1 = P_1O_2M = \tfrac{1}{2}V - O_1;$$

$$OO_1 = R_1 - R, \quad \text{and} \quad O_1O_2 = R_1 - R_2.$$

Hence

$$\frac{O_1O_2}{O_1O} = \frac{R_1 - R_2}{R_1 - R} = \frac{\sin \frac{1}{2}V}{\sin(\frac{1}{2}V - O_1)}. \quad . \quad . \quad . \quad (17)$$

From this we find

$$R_2 = R_1 - (R_1 - R)\frac{\sin \frac{1}{2}V}{\sin(\frac{1}{2}V - O_1)}. \quad . \quad . \quad (18)$$

We may assume values for any two of the quantities R_1, R_2, and O_1, and find the other from either of the above equations.

We observe that $O_1 + O_2 = \frac{1}{2}V$, which is known. Hence if a value of O_1 is assumed, or if O_1 is computed from R_1 and R_2, O_2 also becomes known. There are then but three independent quantities, R_1, R_2, and O_1 (or O_2).

When this is used as a transition curve, R_1 is much greater than R; the first branch is short, or O_1 is small; hence

$$\frac{R_1 - R_2}{R_1 - R} = \frac{\sin \frac{1}{2} V}{\sin (\frac{1}{2} V - O_1)}$$

does not much exceed unity; or $R_1 - R_2$ does not much exceed $R_1 - R$, or R_2 is not much less than R. Hence, by sharpening the main portion of the curve a small amount, the ends of the curve may be much flattened.

To determine the distance MM' between the middle points of a simple curve and a three-centered compound curve joining the same tangent points. (Fig. 81.)

From the triangle OO_1O_2,

$$OO_2 = O_1O_2 \frac{\sin O_1}{\sin \frac{1}{2} V} = (R_1 - R_2) \frac{\sin O_1}{\sin \frac{1}{2} V}.$$

Then $\quad MM' = OO_2 + O_2M' - OM$

$$= (R_1 - R_2) \frac{\sin O_1}{\sin \frac{1}{2} V} - (R - R_2). \quad . \quad . \quad . \quad (19)$$

Problem.—To connect two tangents by a compound curve of four branches, the first two consuming one half of the vertex angle V.

Let $A'P_1mP_2B'$ represent the curve.

Let $\quad A'O_1 = R_1,$

$P_1O_2 = R_2.$

The vertex distance

$Vm = E,$

and $\quad A'O_1P_1 = O_1.$

Draw O_2H and O_2K perpendicular to $A'V$ and to $A'O_1$ respectively. Then

$A'K = HO_2 = VO_2 \cos \frac{1}{2} V$

$= (E + R_2) \cos \frac{1}{2} V.$

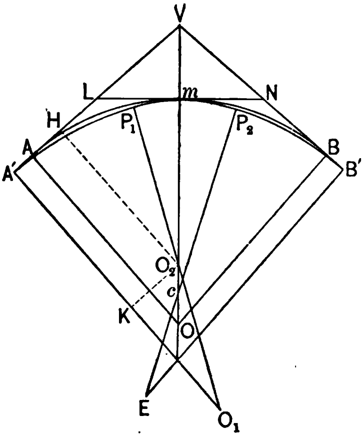


FIG. 82.

Hence

$$\cos O_1 = \frac{KO_1}{O_1O_2} = \frac{A'O_1 - A'K}{O_1O_2} = \frac{R_1 - (E + R_2)\cos \frac{1}{2}V}{R_1 - R_2},$$

and

$$\operatorname{vers} O_1 = 1 - \cos O_1 = \frac{E\cos \frac{1}{2}V - R_2 \operatorname{vers} \frac{1}{2}V}{R_1 - R_2}. \tag{20}$$

Transposing, we have

$$R_1 = \frac{E\cos \frac{1}{2}V - R_2(\cos O_1 - \cos \frac{1}{2}V)}{\operatorname{vers} O_1},$$

and
$$R_2 = \frac{E\cos \frac{1}{2}V - R_1 \operatorname{vers} O_1}{\cos O_1 - \cos \frac{1}{2}V}.$$

Also, $HV = VO_2 \sin \frac{1}{2}V = (E + R_2)\sin \frac{1}{2}V;$

$$A'H = KO_2 = O_1O_2 \sin O_1 = (R_1 - R_2)\sin O_1.$$

Hence
$$A'V = T_1 = (E + R_2)\sin \frac{1}{2}V + (R_1 - R_2)\sin O_1. \tag{21}$$

Since $\cot \frac{1}{2}O_1 = \dfrac{\sin O_1}{\operatorname{vers} O_1}$, we have, from (20) and (21),

$$\cot \frac{1}{2}O_1 = \frac{\sin O_1}{\operatorname{vers} O_1} = \frac{T_1 - (E + R_2)\sin \frac{1}{2}V}{E\cos \frac{1}{2}V - R_2 \operatorname{vers} \frac{1}{2}V}. \tag{22}$$

In the figure the curve is flattened at the ends, but either or both ends may be sharpened, the equations applying without change.

Supposing the vertex distance E to be known, we may assume R_1 and R_2 and find O_1 from (20), then T_1 from (21).

Again, we may assume R_2 and T_1, and find O_1 from (22), then R_1 from (20).

Again, we may assume R_2 and O_1, and find R_1 from (20), then T_1 from (21).

Finally, we may assume T_1 and O_1, and find R_2 from (22), then R_1 from (20).

After finding the unknown quantities, we may lay out the curve $A'P_1m$, consuming one half of the vertex angle V. Then, letting $B'E = R_1$, $P_2C = R_2$, $B'V = T_1$, and $B'EP_2 = O_1$, we may assume values without reference to the corresponding values in the curve $A'P_1m$; and find and lay out a curve mP_2B', differing to any possible extent from the curve $A'P_1m$.

Otherwise, drawing the tangent LmN (Fig. 82), we have

$$mLV = mNV = \tfrac{1}{2}V, \quad mL = T'' = E \cot \tfrac{1}{2}V; \quad LV = \frac{E}{\sin \frac{1}{2}V},$$

and
$$A'L = T' = A'V - LV = T_1 - \frac{E}{\sin \frac{1}{2}V}$$

With the vertex angle $mLV = \frac{1}{2}V$ and the tangents $A'L$ and Lm a compound curve $A'P_1m$ of two branches may be located as usual, using eqs. (1) and (2). Similarly for the curve mB.

It is evident that a compound curve of any number of branches and having in general any radii may connect two fixed tangent points. We may, for example, lay out all but two of the proposed number of branches ending, we will suppose, at m, Fig. 82. Then run out and measure or compute mN; likewise NB'. Then observing that the vertex angle at N is equal to V less the central angles already run off, we may connect the tangent points m and B' as usual, thus completing the curve $A'mB'$.

Let us suppose the compound curve to pass through the middle point of the simple curve AmB, Fig. 82, of radius $AO = R$, connecting the tangents AV and BV.

Then
$$mV = E = R(\sec \tfrac{1}{2}V - 1),$$
or
$$E \cos \tfrac{1}{2}V = R(1 - \cos \tfrac{1}{2}V).$$

Multiplying by $\dfrac{\sin \frac{1}{2}V}{\cos \frac{1}{2}V} = \tan \frac{1}{2}V$, we obtain

$$E \sin \tfrac{1}{2}V = R(\tan \tfrac{1}{2}V - \sin \tfrac{1}{2}V).$$

Now since these equations express the relations existing among the elements R, V, and E of a simple curve, and since, too, eqs. (20) and (21) express the relations among the elements of a compound curve, it is plain that if we substitute in eqs. (20) and (21) the value of E (or of $E \sin \frac{1}{2}V$ and of $E \cos \frac{1}{2}V$) drawn from these equations, the resulting equations will express the relations among the elements of a compound curve that will pass through the middle point m of a simple curve of radius R and central angle V.

We thus find

$$\left.\begin{aligned} \text{vers } O_1 &= \frac{(R - R_2) \text{ vers } \frac{1}{2}V}{R_1 - R_2}, \\ \text{or} \qquad R_1 &= \frac{(R - R_2) \text{ vers } \frac{1}{2}V}{\text{vers } O_1} + R_2, \\ \text{or} \qquad R_2 &= \frac{R \text{ vers } \frac{1}{2}V - R_1 \text{ vers } O_1}{\text{vers } \frac{1}{2}V - \text{vers } O_1}. \end{aligned}\right\} \quad . \quad . \quad (23)$$

$$\text{Also,} \quad T_1 = R \tan \tfrac{1}{2}V + (R_1 - R_2) \sin O_1 - (R - R_2) \sin \tfrac{1}{2}V. \quad (24)$$

$$\text{But} \quad T_1 = VA + AA' = R \tan \tfrac{1}{2}V + AA'.$$

Hence

$$AA' = d = (R_1 - R_2) \sin O_1 - (R - R_2) \sin \tfrac{1}{2}V. \quad . \quad . \quad (25)$$

These results are of course easily deduced from the figure in the same way as results were derived from Fig. 69 and others.

For example,

$$AA' = KO_2 - AH = (R_1 - R_2) \sin O_1 - (R - R_2) \sin \tfrac{1}{2}V,$$

and

$$T_1 = A\,V + AA' = R \tan \tfrac{1}{2}V + (R_1 - R_2) \sin O_1 - (R - R_2, \sin \tfrac{1}{2}V.$$

Example.—Let $O_1 = 24° 15'$, $R_1 = 905$, and $V = 81°$.

It is required to find the radius $P_1O_2 = R_2$ of the second branch of the compound curve passing through the middle point of a simple curve of radius $R = 630$ (Fig. 82). We have

$$E \cos \tfrac{1}{2}V = R\,(\sec \tfrac{1}{2}V - 1) \cos \tfrac{1}{2}V = R(1 - \cos \tfrac{1}{2}V) = R \text{ vers } \tfrac{1}{2}V$$

$$= 630 \times .23959 = 150.94;$$

$$R_1 \text{ vers } O_1 = 905 \times .08824 = 79.86.$$

Also,

$$\cos O_1 - \cos \tfrac{1}{2}V = .15136.$$

Then

$$R_2 = 71.08 \div .15136 = 469.6; \qquad O_2 = \tfrac{1}{2}V - O_1 = 16° \, 15'.$$

2. If the radius R_2 is common to both central branches, we have a curve of three branches shown in Fig. 83. The same equations, (20), (21), and (22), apply.

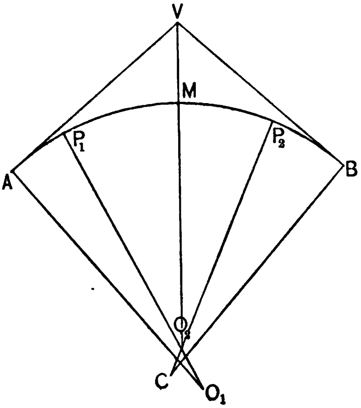


FIG. 83.

Example.—Let

$V = 80° \, 16'$, $O_1 = 12° \, 40'$,

$R_1 = 1011.1$, and $R_2 = 860$.

Then

$$R_1 - R_2 = 151.1.$$

From the first of group (20) we find

$$E + R_2 = \frac{1011.1 - 151.1 \times .9756}{.7645} = 863.7 \div .7645 = 1129.8.$$

$$\therefore \quad E = 1129.8 - 860 = 269.8.$$

(21) gives

$$T_1 = (E + R_2) \sin \tfrac{1}{2}V + (R_1 - R_2) \sin O_1$$

$$= 1129.8 \times .6446 + 151.1 \times .2193 = 728.27 + 33.14 = 761.4.$$

In the curve MP_2B we have

$$E = 269.8, \quad P_2O_2 = R_2 = 860.$$

Let $\qquad BCP_2 = O_1 = 17° \, 50'.$

(20) gives

$$R_1 = \frac{269.8 \times .7645 - 860 \times .1874}{.04805} = \frac{45.10}{.04805} = 938.6.$$

Also, from (21),

$$BV = T_1 = 1129.8 \times .6446 + 151.1 \times .3062 = 728.27 + 46.27$$
$$= 774.54.$$

3. If the end branches have a common radius, R_1, and central-angle, O_1, and the central branches a common radius, R_2, we have a symmetrical curve of three branches shown in Fig. 84. Equations (20), (21), and (22) apply.

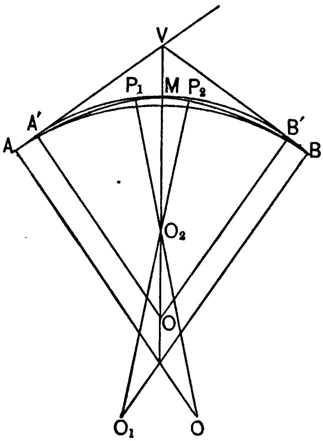


FIG. 84.

I. Suppose the curve is flattened at the ends.

Example.—Let

$$V = 67°, \quad R_1 = 3437.7,$$

and $\quad R_2 = 1432.4.$

It is required that the compound curve shall be tangent at the middle point of the simple curve whose radius is $R = 2291.8$.

We have

$$E = R \text{ exsec } \tfrac{1}{2}V = 2291.8 \times 0.1992 = 456.53.$$

Eq. (20) gives

$$\cos O_1 = \frac{R_1 - (E + R_2)\cos\frac{1}{2}V}{(R_1 - R_2)} = \frac{3437.7 - 1575.1}{2005.3} = .92884.$$

$$\therefore \quad O_1 = 21° \, 45' \quad \text{and} \quad O_2 = V - 2O_1 = 23° \, 30'.$$

Now, from (21),

$$T_1 = (E + R_2)\sin\tfrac{1}{2}V + (R_1 - R_2)\sin O_1$$
$$= 1888.9 \times .55194 + 2005.3 \times .37056 = 1042.56 + 743.08$$
$$= 1785.64.$$

The tangent of the simple curve is

$$A'V = R \tan \tfrac{1}{2}V = 1516.92.$$

Hence the tangent point of the compound curve is back of the tangent point of the simple curve 268.72 feet.

Example 2.—Let $V = 60°$, $R = 2000$, $R_2 = 1200$, and $AA' = d = 240$. To find E, T_1, O_1, and R_1.

$$E = R \text{ exsec } \tfrac{1}{2}V = 2000 \times .1547 = 309.4. \quad \therefore \; E + R_2 = 1509.4.$$

$$T_1 = d + R \tan \tfrac{1}{2}V = d + 2000 \times .57735 = 240 + 1154.7 = 1394.7.$$

$$\cot \tfrac{1}{2}O_1 = \frac{1394.7 - 1509.4 \times .5000}{309.4 \times .8860 - 1200 \times .134} = \frac{640}{107.18} = 5.97126.$$

$$\therefore \; O_1 = 19°.$$

Now (20) gives

$$R_1 = \frac{267.94 - 1200 \times .0795}{.05448} = \frac{172.54}{.05448} = 3167.0.$$

Observe that $E \cos \tfrac{1}{2}V = 267.94$ is computed in finding $\cot \tfrac{1}{2}O_1$. If O_1 is given, we have, from (23),

$$R_1 = \frac{(R - R_2) \text{ vers } \tfrac{1}{2}V}{\text{vers } O_1} + R_2.$$

Then find T_1 (and E if desired) as before.

II. The curve sharpened at the tangents.

This will occur when the compound curve is a moderately easy connecting curve between two tracks, the curves being sharpened at the ends to fit the frogs.

Example 1.—Let $V = 27° 40'$, and $E = 45$. Assume $R_1 = 744$, and $O_1 = 6° 34'$. Eq. (20) gives

$$R_2 = \frac{43.695 - 4.881}{.022444} = 1729.4.$$

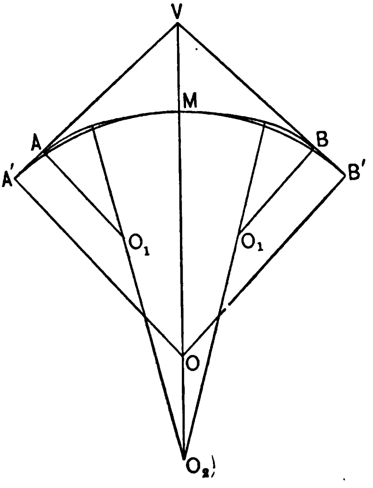


Fig. 85.

Eq. (21) gives

$$T_1 = 1774.4 \times .2391 - 985.4 \times .11436 = 424.26 - 112.69 = 311.6.$$

Example 2.—Let $V = 63°$, and $E = 270$. Assume $R_1 = 744$, and $O_1 = 20° 31'$.

Eq. (20) gives

$$R_2 = \frac{270 \times .85264 - 744 \times .06333}{.08403} = \frac{183.10}{.08403} = 2178.9.$$

Eq. (21) gives

$$AV = T_1 = 2448.9 \times .5225 + 1434.9 \times .35021$$
$$= 1279.6 + 502.5 = 1782.1.$$

Problem.—To draw a tangent to two given curves.

Let AP_1 and BP_2 be the curves, $AO = R_1$, and $BO' = R_2$.

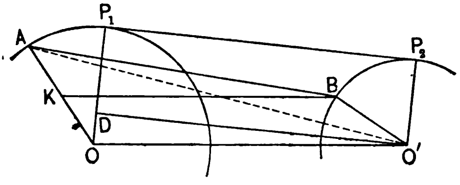


FIG. 86.

I. When the curves are visible from each other.

In this case a point P_1 may be found by trial on one of the curves, through which if a tangent line be drawn, it will be tangent to the other curve also.

II. When the curves are not visible from each other.

(A) Suppose the distance between the centers $OO' = c$, and the direction of OO' to be known. Let $DOO' = O$.

Run any convenient radius AO. Then the angle AOO' is known. Let $P_1P_2 = a$ represent the tangent line, and suppose $O'D$ to be drawn parallel to it. Then

$$\cos O = \frac{DO}{O'O} = \frac{R_1 - R_2}{c}.$$

Also, $P_1P_2 = DO' = c \sin O.$

Then $AOP_1 = AOO' - DOO',$

which gives the length of the arc AP_1, and consequently the position of P_1.

P_1P_2 makes an angle with the tangent at A equal to AOP_1, and therefore P_1P_2 is determined in every respect. If more convenient, run any line AB, and measure the angle $OAB = A$. Suppose BK to be drawn parallel to OO', and let $ABK = B$. We have

$$\sin B = \sin A\frac{R_1 - R_2}{c}.$$

Then $AOO' = AKB = 180° - (A + B).$

Now find O, AOP_1, and $P_1P_2 = a$ as before.

(B) Suppose that OO' is not known.

Run and measure any line $AB = d$, and the angle $BAK = A$. Then, in the triangle ABK, we have

$$AB = d, \quad AK = R_1 - R_2, \quad \text{and} \quad KAB = A.$$

Hence, Chap. III, formula (8),

$$\tan B = \frac{\sin A}{\frac{AB}{AK} - \cos A} = \frac{\sin A}{\frac{d}{R_1 - R_2} - \cos A}$$

Then $c = BK = AK\frac{\sin A}{\sin B} = (R_1 - R_2)\frac{\sin A}{\sin B}.$

Now proceed as before.

We have already considered generally the problem of connecting two straight lines by a compound curve of two or more branches.

We now come to consider the problem of connecting two curves by a *third* curve, thus forming a compound curve of three branches.

Problem.—To locate a curve tangent internally to two given curves, as shown in Fig. 87.

Let $AO_1 = R_1$, and $CO_2 = R_2$, represent the radii of the curves.

I. Suppose the tangent point on either curve to be given. Let A be the point.

By construction. Make $An = CO_2 = R_2$. Draw nO_2 and at the middle point of nO_2 erect a perpendicular to it to meet AO_1 in O. O is the center and $AO = R$ is the radius of the connecting curve.

Calculation.—Run and measure a line from A to any point C on the other curve. Let $AC = a$.

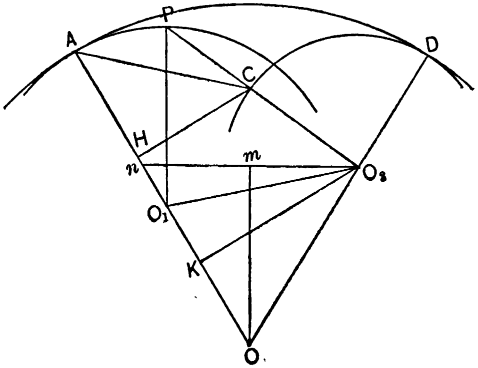


FIG. 87.

Measure the angles $PCA = C$, and $CAO = A$ (P is on CO_2 prolonged).

Draw the perpendiculars CH and O_2K to AO_1.

Evidently the angle between CO_2 and AO_1 is equal to $A - C$. Therefore

$$AK = AH + HK = a \cos A + R_2 \cos (A - C).$$

Similarly, $$KO_2 = a \sin A + R_2 \sin (A - C).$$

Now $Kn = AK - An = AK - R_2$. Let $KnO_2 = n$. Then

$$\tan n = \frac{KO_2}{Kn}; \quad nO_2 = \frac{Kn}{\cos n}; \quad nO = \frac{\frac{1}{2} nO_2}{\cos n} = \frac{Kn}{2 \cos^2 n} = \frac{Kn}{1 + \cos 2n}.$$

Finally, $$AO = DO = R_2 + nO = R.$$

If ACO_2 can be made a straight line,

$$C = 0,\ AK = (a + R_2)\cos A,\quad \text{and}\quad KO_2 = (a + R_2)\sin A.$$

We have $nOO_2 = 180° - 2n$, which gives the central angle, and consequently the length of the connecting curve.

To find $O_1O_2 = c$: Find KO_2 and AK as above. Then

$$KO_1 = AK - R_1;\quad \tan KO_2O_1 = \frac{KO_1}{KO_2},\quad \text{and}\quad O_1O_2 = c = \frac{KO_2}{\cos KO_2O_1}.$$

If the tangent point D on the curve of shorter radius is given, lay off on DO_2 prolonged a distance $Dn = AO_1 = R_1$, and proceed precisely as above.

II. Suppose the radius of the connecting curve to be given. Consider any radius of either of the curves that will, when prolonged, intersect the other curve. Let O_2C represent such a radius, and O_2CP the same line prolonged.

Measure $CP = b$, and $O_1PO_2 = P$. Let $PO_1O_2 = O_1$.

Now $$\tan O_1 = \frac{\sin P}{\dfrac{PO_1}{PO_2} - \cos P} = \frac{\sin P}{\dfrac{R_1}{b + R_2} - \cos P}.$$

(See Chap. III, formula (8).)

Then

$$PO_2O_1 = 180° - (P + O_1).\quad \text{Also,}\quad O_1O_2 = c = R_1\frac{\sin P}{\sin O_2}.$$

The triangle OO_1O_2 gives

$$\cos OO_1O_2 = \frac{(R - R_1)^2 + c^2 - (R - R_2)^2}{2c(R - R_1)},$$

or $$\cos OO_1O_2 = \frac{(c + R - R_1)^2 - (R - R_2)^2}{2c(R - R_1)} - 1.$$

Then $$AO_1P = 180° - (PO_1O_2 + OO_1O_2).$$

This gives the distance from P to the tangent point A.

Also,

$$\tan O_1OO_2 = \frac{\sin OO_1O_2}{\frac{OO_1}{O_1O_2} - \cos OO_1O_2} = \frac{\sin OO_1O_2}{\frac{R - R_1}{c} - \cos OO_1O_2},$$

which gives the length of the required curve.

If C is at the intersection of the curves, $b = 0$; otherwise the same formulas apply.

The above solution applies whatever may be the relative position of the given curves.

When, however, the given curves are tangent internally there is no connecting curve.

Problem.—To locate a given curve tangent externally to two given curves, as shown in Fig. 88.

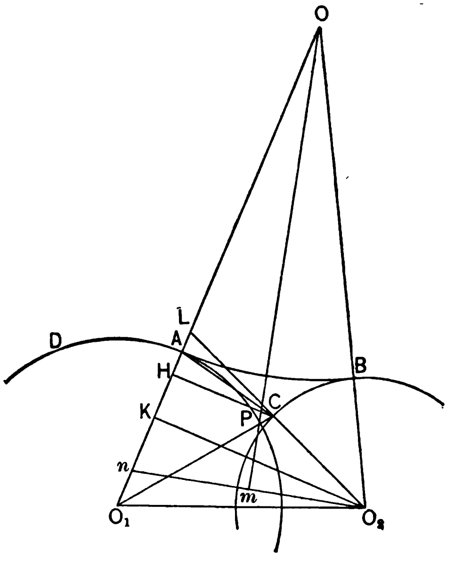


FIG 88.

Let A be the given tangent point. Let $AO_1 = R_1$, and $BO_2 = R_2$.

Make $An = BO_2$, and at the middle point of nO_2 erect a perpendicular to nO_2 to meet AO_1, prolonged in O. O is the center, and $AO = R$ is the radius, of the required curve.

Calculation.—Run the line $AC = a$, as in the preceding problem, and suppose the perpendiculars CH and O_2K to be drawn. Let $ACL = C$, and $CAO_1 = A$. Then

$$AK = AH + HK = a \cos A + R_2 \cos (A - C),$$

and $\quad KO_2 = a \sin A + R_2 \sin (A - C).$

Now $\quad Kn = An - AK = R_2 - AK.$

Let $KnO_2 = n$. Then

$$\tan n = \frac{KO_2}{Kn}; \qquad nO_2 = \frac{Kn}{\cos n};$$

$$nO = \frac{\frac{1}{2}nO_2}{\cos n} = \frac{Kn}{2 \cos^2 n} = \frac{Kn}{1 + \cos 2n}.$$

Finally, $\quad AO = nO - An = nO - R_2.$

Suppose the radius of the connecting curve $AO = R$ to be given, and the position of the tangent points and the length of the curve required.

Draw CO_1 intersecting the curve AD in P. Measure $CP = b$, and the angle $O_1CO_2 = C$. Let $CO_1O_2 = O_1$. Now

$$\tan O_1 = \frac{\sin C}{\frac{CO_1}{CO_2} - \cos C} = \frac{\sin C}{\frac{b + R_1}{R_2} - \cos C}. \quad \text{(Chap. III, eq. (8)).}$$

Then $\quad CO_2O_1 = 180° - (C + O_1).$

Also, $\quad O_1O_2 = c = R_2 \dfrac{\sin C}{\sin O_1}.$

The triangle OO_1O_2 gives

$$\cos OO_1O_2 = \frac{(R + R_1)^2 + c^2 - (R + R_2)^2}{2c(R + R_1)},$$

or $$\cos OO_1O_2 = \frac{(c + R + R_1)^2 - (R + R_2)^2}{2c(R + R_1)} - 1.$$

Then $\quad AO_1P = OO_1O_2 - CO_1O_2.$

This gives the position of the tangent point A with reference to P.

The above solution applies whatever the relative position of the given curves may be.

When, however, the given curves are tangent internally, or one of them, prolonged at pleasure, lies entirely within the other, there is no connecting curve.

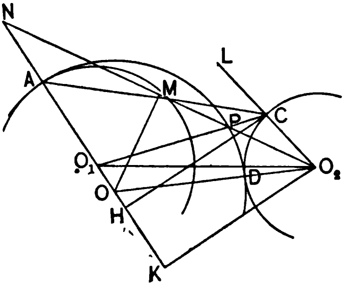


FIG. 89.

Problem. — To locate a curve tangent to two given curves, externally to one and internally to the other, thus forming a compound and a reverse curve.

Let $AO_1 = R_1$, $CO_2 = R_2$, $AO = R$.

Let A be the given tangent point.

By Construction.—On the radius O_1A *prolonged* make $An = CO_2 = R_2$. Draw nO_2, and at the middle point of nO_2 erect a perpendicular mO to meet AO_1 in O.

O is the center and $AO = DO = R$ is the radius of the required curve.

Calculation.—Run the line $AC = a$, as in the preceding problems, and suppose the perpendiculars CH and O_2K to be drawn.

Let $ACL = C$, and $CAO_1 = A$.
Then $AK = AH + HK = a \cos A + R_2 \cos (A - C)$,
and $KO_2 = a \sin A + R_2 \sin (A - C)$.

Now $Kn = An + AK = R_2 + AK$. Let $KnO_2 = n$.
Then

$$\tan n = \frac{KO_2}{Kn};\quad nO_2 = \frac{Kn}{\cos n};\quad nO = \frac{\frac{1}{2}nO_2}{\cos n} = \frac{Kn}{2\cos^2 n} = \frac{Kn}{1 + \cos 2n}.$$

Finally, $$AO = nO - An = nO - R_2.$$

Suppose the radius of the connecting curve $AO = R$ to be given, and the position of the tangent points and the length of the connecting curve required. Draw CO_1 intersecting the curve

AD in P. Measure $CP = b$ and the angle $O_1CO_2 = C$. Let $CO_1O_2 = O_1$. Now, Chap. III, formula (8),

$$\tan O_1 = \frac{\sin C}{\dfrac{CO_1}{CO_2} - \cos C} = \frac{\sin C}{\dfrac{b + R_1}{R_2} - \cos C}.$$

Then $$CO_2O_1 = 180° - (C + O_1).$$

Also, $$O_1O_2 = c = R_2\frac{\sin C}{\sin O_1}.$$

The triangle OO_1O_2 gives

$$\cos OO_1O_2 = \frac{(R - R_1)^2 + c^2 - (R + R_2)^2}{2c(R - R_1)},$$

or $$\cos OO_1O_2 = \frac{(c + R - R_1)^2 - (R + R_2)^2}{2c(R - R_1)} - 1.$$

Then $$AO_1P = 180° - (CO_1O_2 + OO_1O_2).$$

This gives the position of the tangent point A with reference to P.

The above solution applies for all positions of the given curves. When, however, the curves are tangent externally there is no connecting curve.

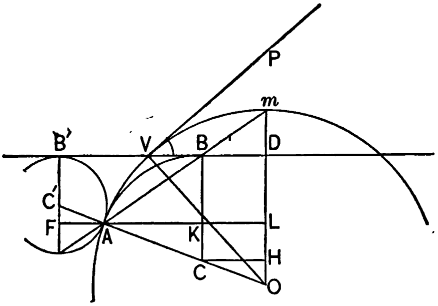


FIG. 90.

Problem.—To find a curve tangent to a given curve, and to a straight line, which intersects the curve. (Fig. 90.)

A. Let $AO = R'$ = the radius of the given curve AVm, and let BD be the line.

The radius R of the required curve may be given, or the tangent point A on the curve, or the tangent point B on the line.

I. Given the radius $AC = R$. Draw VO. Suppose $FAKL$ and CH drawn parallel to BD. Measure the angle PVD (which is equal to DOV) and represent it by A.

Let $ACB = AOD = O$, and $Dm = R'$ vers $A = a$.

Now $$mL - BK = Dm.$$

That is, $$R' \text{ vers } O - R \text{ vers } O = a, \quad \text{or vers } O = \frac{a}{R' - R}.$$

$$VB = VD - CH = R' \sin A - (R' - R) \sin O.$$

This gives the position of B.

Also, $$AOV = AOD - VOD = O - A.$$

This gives the position of A. From A or from B the curve may be run.

We have, from above,

$$\sin O = \frac{R' \sin A - VB}{R' - R}.$$

Whence $$\tan \tfrac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{a}{R' \sin A - VB}.$$

II. Suppose the tangent point A to be assumed.

Let $AOV = B$. Then $AOD = A + B = O$ is known.

We have, from above,

$$R = R' - \frac{a}{\text{vers } O}.$$

BV is found as above.

III. Suppose the tangent point B to be assumed. Let $VB = b$. Substituting this value of VB in the value of $\tan \frac{1}{2}O$, we have

$$\tan \tfrac{1}{2}O = \frac{a}{R' \sin A - b}.$$

Then $$R = R' - \frac{a}{\text{vers } O}.$$

Now O and R give the length of the arc AB and the tangent point A.

Draw ABm, which is a straight line as shown above.

$$\text{Now} \qquad DBm = ABV = \tfrac{1}{2}ACB = \tfrac{1}{2}O,$$

$$mD = R' \operatorname{vers} A = a, \quad \text{and} \quad BD = VD - VB = R' \sin A - VB.$$

$$\text{Then} \qquad \tan mBD = \frac{mD}{BD}, \quad \text{or} \quad \tan \tfrac{1}{2}O = \frac{a}{R' \sin A - VB},$$

as already shown.

B. Suppose the required curve to be tangent to the given curve externally, as also shown in Fig. 90.

Let the central angle $AC'B' = 180° - O$, and $AC' = r$.

$$\text{Then} \qquad ACB = AOD = O.$$

I. Suppose the radius $AC' = r$ given.

$$\text{We have} \qquad mL - B'C' - C'F = Dm.$$

That is,

$$R' \operatorname{vers} O - r - r \cos O = a, \quad \text{or} \quad R' \operatorname{vers} O - 2r + r \operatorname{vers} O = a.$$

Hence

$$\operatorname{vers} O = \frac{a + 2r}{R' + r}; \quad VB' = B'D - VD = (R' + r) \sin O - R' \sin A.$$

This gives the position of B'.

$$\text{Also,} \qquad AOV = AOD - VOD = O - A.$$

This gives the position of A. From A or B' the curve may be run. B' is also given by the angle $AC'B' = 180° - O$.

$$\text{We have, from above,} \quad \sin O = \frac{VB' + R' \sin A}{R' + r}.$$

$$\text{Whence} \qquad \tan \tfrac{1}{2}O = \frac{\operatorname{vers} O}{\sin O} = \frac{a + 2r}{R' \sin A + VB'}.$$

II. Suppose the tangent point A to be assumed.
Let $AOV = B$. Then

$$AOD = A + B = O, \quad \text{and} \quad AC'B' = 180° - (A + B) = 180° - O$$

is known. Then, from above,

$$r = \frac{R' \text{ vers } O - a}{1 + \cos O}.$$

$B'V$ is found as above.

III. Suppose the tangent point B' to be assumed.

$$\text{Let} \qquad VB' = b'; \qquad r = \frac{R' \text{ vers } O - a}{1 + \cos O}.$$

$$\text{Then} \qquad \tan \tfrac{1}{2}O = \frac{a + 2r}{R' \sin A + b'}.$$

r and O.

Give the length of the arc $B'A$ and the tangent point A.

C. Suppose the required curve to be tangent to the given curve internally, the centers of the two curves being on opposite sides of the line. (Fig. 91.)

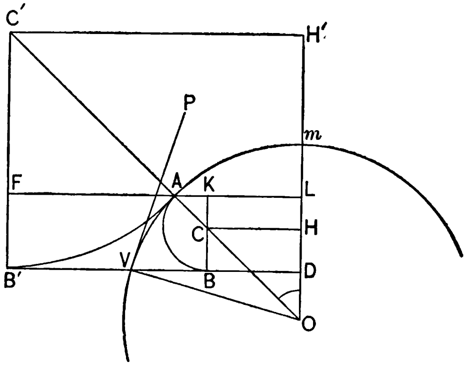


FIG. 91.

Let $DOV = DVP = A$, and $Dm = R'$ vers $A = a$, as before. Let the central angle $ACB = 180 - O$.

Then $$AOD = O.$$

I. Let the radius $AC = r$ be given. Suppose $FAKL$, CH, and $C'H'$ drawn parallel to BD. We have

$$mL + BC + CK = Dm.$$

That is,

R' vers $O + r + r \cos O = a$, or R' vers $O + 2r - r$ vers $O = a$.

Hence $$\text{vers } O = \frac{a - 2r}{R' - r}.$$

Then $$AOV = VOD - AOD = A - O,$$

which gives the position of the tangent point A.

Also, $$VB = VD - CH = R' \sin A - (R' - r) \sin O,$$

which gives the position of B. From A or from B the curve may be run.

From above, $$\sin O = \frac{R' \sin A - VB}{R' - r}.$$

Then $$\tan \tfrac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{a - 2r}{R' \sin A - VB}.$$

II. Suppose the tangent point A is assumed.
Let $AOV = B$. Then $AOD = A - B = O$ is known.
We have, from the above equation,

$$r = \frac{a - R' \text{ vers } O}{1 + \cos O}.$$

Then VB as given above.
III. Suppose the tangent point B to be assumed.
Let $VB = b$. Eliminating the functions of O from the equations involving sin O and vers O, we find

$$r = \frac{A}{2} - \frac{(R' \sin A - b)^2}{4R' - 2a}.$$

Then $\tan \frac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{a - 2r}{R' \sin A - b}$.

r and O give the center C and arc BA and the tangent point A.

D. Suppose the required curve to be tangent to the given curve externally, as also shown in Fig. 91.

Let the central angle $AC'B' = O$.

I. Suppose the radius $AC' = r$ is given.

We have $mL + B'F = Dm$.

That is, $R' \text{ vers } O + r \text{ vers } O = a$.

Hence $\text{vers } O = \frac{a}{R' + r}$.

Then $AOV = VOD - AOD = A - O$,

which gives the position of the tangent point A.

Also,

$$VB' = C'H' - VD = (R' + r) \sin O - R' \sin A,$$

which gives the position of B'.

From A or from B' the curve may be located.

From above, $\sin O = \frac{R' \sin A + VB'}{R' + r}$.

Then $\tan \frac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{a}{R' \sin A + VB'}$.

II. Suppose the tangent point A is assumed.

Let $AOV = B$. Then $O = AC'B' = AOD = A - B$ is known.

Then equation above gives

$$r = \frac{a}{\text{vers } O} - R'.$$

VB' is found as above.

III. Suppose the tangent point B' is assumed. Let $VB' = b$. We have, from equation above,

$$\tan \frac{1}{2}O = \frac{a}{R' \sin A + b}.$$

Then r is found as above.

r and O give the center C', the arc $B'A$, and the tangent point A.

Also, $$VOA = VOD - AOD = A - O$$

gives the arc VA and the tangent point A.

Problem.—To find a curve tangent to a given curve internally, and to a straight line, that does not intersect the curve.

A. Let $AO = R'$ = radius of given curve Am, and BD = given line.

I. Suppose the radius, $AC=BC=R$, of the required curve AB to be given.

Draw OD perpendicular, and suppose AFK drawn parallel to BD.

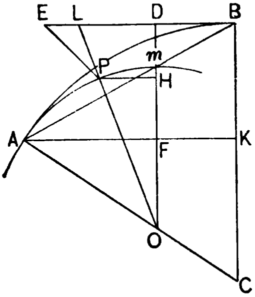


FIG. 92.

Let $$Dm = a,$$

and $$AOF = ACK = O.$$

Now $$BK = Dm + mF,$$

or $$R \text{ vers } O = a + R' \text{ vers } O, \quad \text{or} \quad \text{vers } O = \frac{a}{R - R'}.$$

This gives the length of the arc mA and the tangent point A.

Also, $$DB = FK = (R - R') \sin O.$$

This determines the position of B by giving its distance from D, the point on the line opposite to m.

We have, from above,

$$\sin O = \frac{BD}{R - R'}.$$

Whence $$\tan \tfrac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{a}{BD}.$$

II. If the tangent point A is given, we have $AOm = O$ given, and from the equation above we find

$$R = \frac{a}{\text{vers } O} + R'.$$

Also, $$AK = R \sin O,$$

which gives the position of B.

III. If the tangent point B is assumed, let $BD = b$, and we have, from above,

$$\tan \tfrac{1}{2}O = \frac{a}{b}.$$

We also have, as explained above,

$$\tan \tfrac{1}{2}O = \frac{Dm}{BD} = \frac{a}{b}.$$

Also, $$R = R' + \frac{a}{\operatorname{vers} O}.$$

O and R give the length of the arc BA and the tangent point A.

If Dm is unknown, run and measure any line $EP = b$ and draw PH. Measure EPL (between EP and OP prolonged) and $PEL = E$. Then

$$OLD = E + P, \quad \text{and} \quad POm = O = 90^\circ - (E + P).$$

Hence $$DO = DH + HO = b \sin E + R' \cos O,$$

and $$Dm = DH - mH = b \sin E - R' \operatorname{vers} mOP.$$

B. Let the curve be tangent to a given curve externally, and to a straight line that does not intersect the curve.

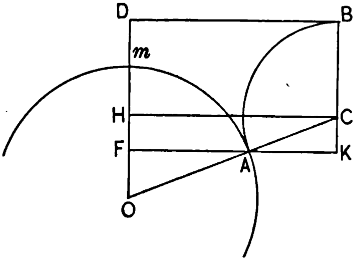


FIG. 93.

Let $AO = R'$ = the radius of the given curve Am, and let BD be the line.

I. Suppose the radius, $AC = BC = r$, of the required curve to be given. Suppose CB and OD drawn perpendicular, and CH and FAK parallel, to BD. The notation is uniform throughout this class of problems.

Now $$DO = Dm + mO = a + R'.$$

Also, $$DO = BC + HO = r + (R' + r)\cos O.$$

Therefore $$a + R' = r + (R' + r)\cos O,$$

or $$\cos O = \frac{a + R' - r}{R' + r}, \quad \text{and} \quad \text{vers}\, O = \frac{2r - a}{R' + r}.$$

This gives the length of the arc *Am* and the tangent point *A*.

Also, $$BD = CH = (R' + r)\sin O.$$

This gives the position of the tangent point *B*.

Also, $$\tan \tfrac{1}{2}O = \text{vers}\, O \div \sin O = \frac{2r - a}{BD}.$$

II. If the tangent point *A* is assumed, $AOH = O$ is known, and the above equation gives

$$r = \frac{a + R'(1 - \cos O)}{1 + \cos O}.$$

Then $$BD = (R' + r)\sin O, \quad \text{as before.}$$

III. Suppose the tangent point *B* is assumed. Let $BD = b$. We have

$$\tan \tfrac{1}{2}O = \frac{2r - a}{b} = \frac{b}{2R' + a}.$$

Observing that $HO = DO - BC = a + R' - r$, $CH = b$, and $CO = R' + r$, we deduce, from the triangle *CHO*, or from the above,

$$r = \frac{a}{2} + \frac{b^2}{2(2R' + a)}.$$

O and *r* give the length of the arc *AB* and the tangent point *A*.

"WYE" PROBLEMS.

The preceding equations apply at once to "wye" problems.

In this case a straight line and two curves are tangent to each other, or three curves are tangent to each other.

Example 1.—A 4° curve, radius $P_1O_1 = R = 1432.4$, ends in a tangent P_1P_2. It is required to locate a curve PP_2 tangent to the straight line and curve. Draw O_2n parallel to P_1P_2.

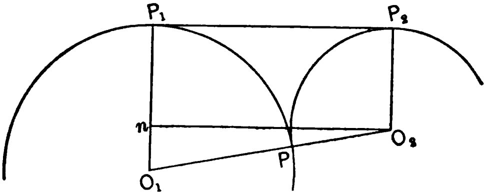


FIG. 94.

Let $nO_1O_2 = O_1$ and $O_1O_2P_2 = O_2$.

I. Assume the radius $O_2P_2 = r = 954.9$ corresponding to a 6° curve.

$$\text{Now} \qquad \cos O_1 = \frac{nO_1}{O_1O_2} = \frac{R-r}{R+r} = \frac{477.5}{2387.3} = .2000.$$

$$\therefore\ O_1 = 78^\circ.46,$$

$$\text{and} \qquad O_2 = 180^\circ - 78^\circ.46 = 101^\circ.54.$$

$$\text{Hence} \qquad \text{arc } PP_1 = \frac{7846}{4} = 1962 \text{ feet},$$

$$\text{and} \qquad \text{arc } PP_2 = \frac{10154}{6} = 1692 \text{ feet}.$$

Also, $P_1P_2 = nO_2 = (R + r)\sin O_1 = 2339.1$ feet.

II. Assume the tangent point P on the curve P_1P.

Then O_1 is given, and, from equation above, we have

$$r = \frac{R(1 - \cos O_1)}{1 + \cos O_1}.$$

$$\text{Also,} \qquad P_1P_2 = (R + r)\sin O_1.$$

This determines the tangent point P_2.

III. Assume the tangent point P_2 on the line P_1P_2.

Let $P_1P_2 = c$. Then, in the triangle nO_1O_2,

$$(R + r)^2 - (R - r)^2 = c^2,$$

or

$$4Rr = c^2, \quad \text{or} \quad r = \frac{c^2}{4R}.$$

O_1 and O_2 and the position of P_1 are now found as in case I.

Observing that the radii are inversely as the degree of the curve, we may substitute the reciprocals of the degrees of the curves in place of the radii in the formulas. Thus

$$\cos O_1 = \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{4} + \frac{1}{6}} = \tfrac{1}{5} = .2000. \quad \therefore\ O_1 = 78^\circ.46.$$

Example 2.—A 2° curve, radius $P_1O_1 = R_1 = 2864.8$, is tangent at P to a 3° curve, radius $PO_2 = R_2 = 1909.9$.

It is required to locate a 0° 55′ curve, radius $P_1O = R = 6250.4$, tangent to both curves.

In the triangle O_1O_2O,

$$O_1O_2 = R_1 + R_2,$$
$$O_1O = R - R_1,$$

and $$O_2O = R - R_2.$$

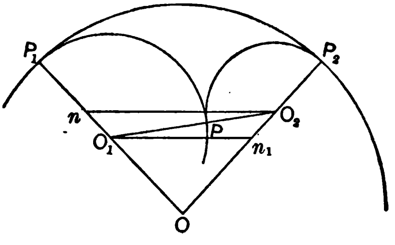


FIG. 95.

Hence the angles may be found by the general formula

$$\cos a = \frac{b^2 + c^2 - a^2}{2bc}.$$

Or, observing that $55' = \frac{11}{12}$ of 1°, we have, as explained above,

$$\cos O = \frac{(\frac{12}{11} - \frac{1}{2})^2 + (\frac{12}{11} - \frac{1}{3})^2 - (\frac{1}{3} + \frac{1}{2})^2}{2(\frac{12}{11} - \frac{1}{2})(\frac{12}{11} - \frac{1}{3})}.$$

Or, multiplying through by 2 × 3 × 11, we have

$$\cos O = \frac{(39)^2 + (50)^2 - (55)^2}{2 \times 39 \times 50} = \frac{83}{325} = .25538.$$

$$\therefore\ O = 75^\circ.2.$$

By simply transposing this equation, we have

$$\cos O_2 = \frac{(50)^2 + (55)^2 - (39)^2}{2 \times 50 \times 55} = .728. \quad \therefore \quad O_2 = 43^\circ.28.$$

Or O_2 could be found by the familiar "sine proportion." Now

$$O_1 = 180^\circ - (O + O_2) = 61^\circ.52, \quad 180^\circ - O_2 = 136^\circ.72,$$

and

$$180^\circ - O_1 = 118^\circ.48.$$

Hence

$$P_1P_2 = 7520 \div \tfrac{11}{12} = 8204 \text{ feet};$$

$$PP_1 = 11848 \div 2 = 5924 \text{ feet};$$

$$PP_2 = 13672 \div 3 = 4557 \text{ feet}.$$

II. Suppose P_1O_1P is assumed. This gives the position of P_1. Then $OO_1O_2 = O_1$ is known. Draw O_2n, making $P_1n = P_2O_2$.

Let $OnO_2 = OO_2n = n$. Then

$$\tan n = \frac{\sin O_1}{\cos O_1 + \dfrac{nO_1}{O_1O_2}} = \frac{\sin O_1}{\cos O_1 + \dfrac{R_1 - R_2}{R_1 + R_2}}.$$

Now

$$nO_2O_1 = O_1 - n, \quad OO_2n = n.$$

$$\therefore \; OO_2P = 2n - O_1, \text{ and } PO_2P_2 = 180^\circ - OO_2P = 180^\circ - (2n - O_1).$$

This gives the length of the arc PP_2 and the position of P_2.

$$O_1OO_2 = 180 - 2n.$$

This gives the length of the arc P_1P_2.

To find R, we have

$$nO_2 = O_1O_2\frac{\sin O_1}{\sin n} = (R_1 + R_2)\frac{\sin O_1}{\sin n};$$

$$nO = \frac{\tfrac{1}{2}nO_2}{\cos n} = \frac{(R_1 + R_2)\sin O_1}{\sin 2n}.$$

Finally,

$$P_1O = R = R_2 + nO.$$

If the tangent point P_2 is assumed, draw O_1n_1, making $P_2n_1 = P_1O_1$, and proceed as above.

Example 3.—A 4° curve, radius $PO_1 = R_1$, is tangent externally to a 5° curve, radius $PO_2 = R_2$. It is required to locate an 8° curve, radius $P_1O = R$, tangent to both curves.

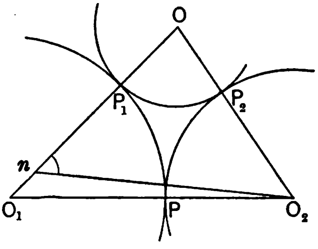


FIG. 96.

We have

$$\cos O = \frac{(\frac{1}{8}+\frac{1}{4})^2+(\frac{1}{8}+\frac{1}{5})^2-(\frac{1}{4}+\frac{1}{5})^2}{2(\frac{1}{8}+\frac{1}{4})(\frac{1}{8}+\frac{1}{5})}$$

$$= \frac{225+169-324}{2\times15\times13} = \frac{7}{39} = .17949.$$

Therefore

$$O = 79° \, 40' = 79°.67;$$

Similarly, $$\cos O_1 \doteq \frac{225+324-169}{2\times15\times18} = \frac{19}{27} = .70370.$$

$$\therefore \; O_1 = 45° \, 17' = 45°.28.$$

Or O_1 may be found by the "sine proportion." Then

$$O_2 = 180° - (O + O_1) = 55° \, 03' = 55°.05.$$

II. Suppose O_1 or O_2 is assumed; O_1, for example. This gives the position of P_1. Draw O_2n, making $P_1n = P_2O_2$.

Let $OnO_2 = OO_2n = n$. Then

$$\tan n = \frac{\sin O_1}{\cos O_1 - \frac{R_1 - R_2}{R_1 + R_2}}.$$

Then

$$nO_2O_1 = n - O_1, \quad \text{and} \quad OO_2O_1 = O_2 = nO_2O_1 + n = 2n - O_1.$$

This gives the arc PP_2 and the position of P_2.

Also, $$O = 180° - (O_1 + O_2),$$

which gives the length of the arc P_1P_2.

To find R, we have

$$nO_2 = O_1O_2\frac{\sin O_1}{\sin n} = (R_1 + R_2)\frac{\sin O_1}{\sin n};$$

$$nO = \frac{\frac{1}{2}nO_2}{\cos n} = \frac{(R_1 + R_2)\sin O_1}{2\sin n \cos n} = \frac{(R_1 + R_2)\sin O_1}{\sin 2n}.$$

Then $$OP_1 = R = nO - nP_1 = nO - R_2.$$

Problem. — Having located a simple curve APB, of given radius $AO = R$, tangent to a given line AG at A, it is required to run a compound curve, $A'P'B'$, of two branches exterior to the simple curve, the first branch having the shorter radius and ending in a given parallel tangent $A'G'$, the second branch being concentric with the simple curve and at a given distance from it.

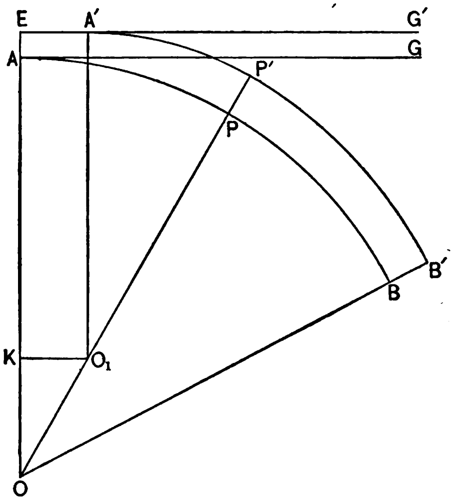


FIG. 97.

To find the angle of the first branch. Let $A'O_1 = R_1$, $BB' = PP' = D$, and $AE = d$. Let $AOP = A'O_1P = O$. Draw O_1K parallel to AG.

Now $KO + R_1 = R + d$, or $KO = R - R_1 + d$.

Also, $O_1O = OP' - O_1P' = R + D - R_1$.

Hence $$\cos O = \frac{KO}{OO_1} = \frac{R - R_1 + d}{R - R_1 + D},$$

or $$\text{vers } O = \frac{D - d}{R - R_1 + D}.$$

This solves the problem fully.

The formula shows that d must be less than D, since vers O must be greater than zero; and that the less is the difference between d and D, the shorter is the first branch, $A'P'$, of the curve.

When $d = D$, vers $O =$ zero; therefore $O =$ zero, the branch $A'P'$ vanishes, and the exterior curve becomes a simple curve, concentric with the curve APB.

Also, $EA' = KO_1 = OO_1 \sin O = (R - R_1 + D) \sin O$,

which gives the position of the tangent point A', and the distance apart, measured on AG, of the head-blocks at A and A'.

When $d =$ zero, AG and $A'G'$ coincide, and the two curves are tangent to the same straight line, at A and at A', as shown in Fig. 98.

The formula for this case becomes

$$\text{vers } O = \frac{D}{R - R_1 + D}.$$

The formula shows that O increases and decreases as R_1 increases and decreases.

Hence, in order that the parallel branch $B'P'$ may be comparatively long, and therefore the branch $A'P'$ correspondingly short, the radius R_1 must be correspondingly short, and the frog at A' sharp.

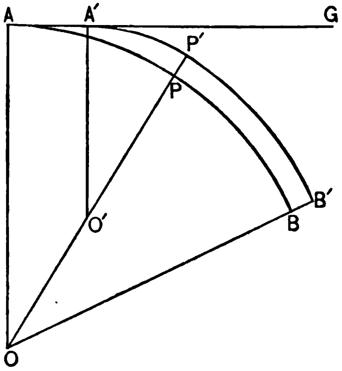


FIG. 98.

Example.—Suppose a number 9 frog placed at A, and a number 7 frog at A'. Let $BB' = D = 22$ feet.

Required the angle $AOP = O$, and the distance AA' between the frogs. In this case

$$AO = R = 744.0, \quad \text{and} \quad A'O_1 = R_1 = 439.4.$$

$$\therefore\ R - R_1 + D = 326.6,$$

whence $$\text{vers } O = \frac{22}{326.6} = .06736, \quad \text{and} \quad O = 21^\circ\ 09' = 21^\circ.15.$$

$$AA' = (R - R_1 + D) \sin O = 326.6 \times .36081 = 117.84.$$

The degree of the curve $A'P'$ is $13^\circ.04$.

$$\therefore\ A'P' = 100\frac{21.15}{13.04} = 162 \text{ feet.}$$

If AA' is given, we have, from the preceding equations,

$$\sin O = \frac{AA'}{R - R_1 + D}, \quad \text{and} \quad \text{vers } O = \frac{D}{R - R_1 + D}.$$

$$\therefore\ \tan \tfrac{1}{2}O = \frac{\text{vers } O}{\sin O} = \frac{D}{AA'}.$$

If any two of the quantities O, D, or AA' are known or assumed, we may find the other from this equation.

We observe that the head-block A' must be placed where the main track, AG, and the first turnout curve, AP are sufficiently far apart to give room for it. This limits the distance AA'.

Problem.—Having located a simple curve, $A'P'B'$, of given radius, $A'O_1 = R_1$, tangent to a given line, $A'G'$, at A', it is required to run a compound curve, APB, of two branches exterior to the simple curve, the first branch having the longer radius and ending in a given parallel tangent, AG; the second branch being concentric with the simple curve and at a given distance from it.

Let $$AO = R,\ A'O_1 = R_1,$$

etc., as before. We have

$$R - KO = R_1 + d, \quad \text{or} \quad KO = R - R_1 - d,$$

and $$OO_1 = OP - O_1P = R - R_1 - D.$$

Hence

$$\cos O = \frac{KO}{OO_1} = \frac{R - R_1 - d}{R - R_1 - D}. \quad \text{vers } O = \frac{d - D}{R - R_1 - D}.$$

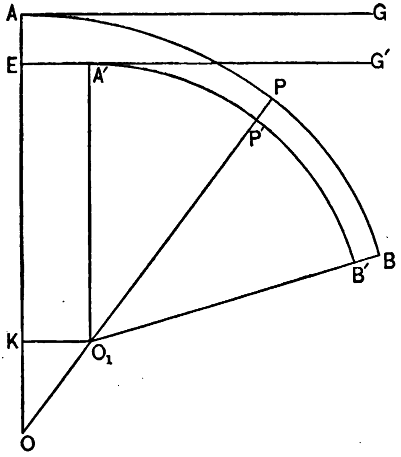


FIG. 99.

Since vers O must be greater than zero, d must be greater than D.

In case $D =$ zero, the arcs BP and $B'P'$ coincide, and the problem is the same as that given under Figs. 69 and 70, and we have

$$\text{vers } O = \frac{d}{R - R_1},$$

as before shown.

In case the simple curve, in the two preceding problems, is exterior to the compound curve, the same solution applies.

These problems are useful about yards and switches when it is desirable or necessary to have turnouts from different switches merge into parallel curves.

The radii R and R_1 are determined by the numbers of the frogs used at A and at A'.

To locate the second branch of a compound or reversed curve from a station A on the first branch of the curve.

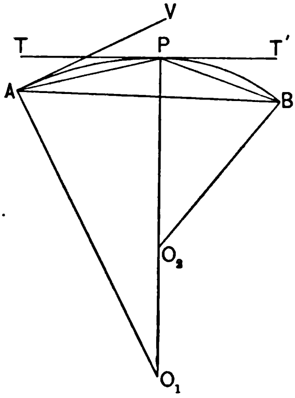


FIG. 100.

AP and PB are the two branches of two curves of D_1 and D_2 degrees; TPT' the common tangent, and AV a tangent at A.

Let n_1 and n_2 be the number of chains in AP and PB respectively. Let l_1 and l_2 represent the long chords AP and PB.

Then the angle

$$APT = \tfrac{1}{2}n_1D_1,$$

and the angle

$$BPT' = \tfrac{1}{2}n_2D_2.$$

$$\therefore \quad APB = 180° - \tfrac{1}{2}(n_1D_1 + n_2D_2).$$

Hence, by formula (8), Chap. III,

$$\tan PAB = \frac{\sin APB}{\frac{AP}{BP} - \cos APB} = \frac{\sin \frac{1}{2}(n_1D_1 + n_2D_2)}{\frac{l_1}{l_2} + \cos \frac{1}{2}(n_1D_1 + n_2D_2)}; \quad . \quad (26)$$

$$BAP + ABP = 180° - APB = \tfrac{1}{2}(n_1D_1 + n_2D_2).$$

Hence, if $n_1D_1 + n_2D_2$ is small, we have, from the above the approximate formula,

$$PAB = \frac{\frac{1}{2}(n_1D_1 + n_2D_2)}{\frac{l_1}{l_2} + 1}$$

$$= \frac{l_2}{l_1 + l_2}\left(\frac{n_1D_1 + n_2D_2}{2}\right) = \frac{n_2}{n_1 + n_2}\left(\frac{n_1D_1 + n_2D_2}{2}\right). \quad . \quad (27)$$

For reversed curves D_2 is negative and the formulas become

$$\tan PAB = \frac{\sin \frac{1}{2}(n_1D_1 - n_2D_2)}{\frac{l_1}{l_2} + \cos \frac{1}{2}(n_1D_2 - n_2D_2)};$$

and
$$PAB = \frac{n_2}{2(n_1 + n_2)}(n_1D_1 - n_2D_2). \quad . \quad . \quad (28)$$

The above formula is easily deduced as follows:

$$BAP + ABP = 180^\circ - APB = APT + BPT' = \frac{n_1D_1 + n_2D_2}{2}.$$

In the triangle ABP the angles are proportional to the opposite sides nearly. Hence

$$BAP = \frac{BP}{AP + BP}(BAP + ABP)$$

$$= \frac{l_2}{l_1 + l_2}\left(\frac{n_1D_1 + n_2D_2}{2}\right) = \frac{n_2}{n_1 + n_2}\left(\frac{n_1D_1 + n_2D_2}{2}\right).$$

Having run a curve with the center O and radius R through a given arc AB, it is required to find the radius of a curve with center O' on AO that will end at C on BO at a given distance A from B.

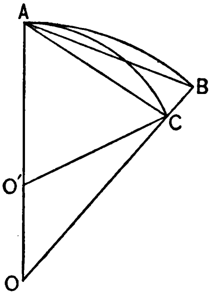


FIG. 101.

Solution.—In the triangle ABC we have

$$BC = a, \quad AB = c = 2R \sin \tfrac{1}{2}O,$$

and $$ABC = B = 90^\circ - \tfrac{1}{2}O.$$

Then $$\tan BAC = \frac{\sin B}{\frac{c}{a} - \cos B}.$$

Now $O' + 2CAO' = 180^\circ$, and $O + 2BAO' = 180^\circ$.

Equating and transposing gives

$$O' - O = 2BAO' - 2CAO' = 2BAC = 2A, \quad \text{or} \quad O' = O + 2A.$$

Hence $$AO' = CO' = CO\frac{\sin O}{\sin O_1} = (R - a)\left(\frac{\sin O}{\sin O_1}\right).$$

Example.—Let $R = 640$, $BC = a = 100$, $O = 40^\circ$. Then $B = 70^\circ$.

Now $c = 1280 \times .3420 = 437.8$; $\frac{c}{a} = 4.378$; $\cos B = .3420$.

Then $\tan A = \dfrac{.9397}{4.036} = .2328. \quad \therefore \quad A = 13° 06',$

and $O' = O + 2A = 66° 12'.$

Finally, $CO = \dfrac{540 \times .6428}{.9205} = 377.1.$

Problem.—Given a compound curve of any number of branches, to find the length of the tangents. Let $AP_1P_2P_3$ (Fig. 102) represent a compound curve of three branches, the central angles being O_1, O_2, and O_3, and the radii R_1, R_2, and R_3. Draw through the centers parallels, and through the tangent points perpendiculars, to the tangent at A.

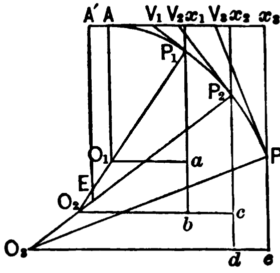


FIG. 102.

$$O_1P_1a = P_1V_1x_3 = O_1,$$
$$O_2P_2c = P_2V_2x_3 = O_1 + O_2,$$

and

$$O_3P_3e = P_3V_3x_3 = O_1 + O_2 + O_3.$$

Let $P_1x_1 = t_1, \quad P_2x_2 = t_2, \quad \text{and} \quad P_3x_3 = t_3.$

We have $t_1 = R_1 \text{ vers } O_1;$

$$t_2 = t_1 + P_1b - P_2c = t_1 + R_2[\cos O_1 - \cos(O_1 + O_2)]$$
$$= R_1 \text{ vers } O_1 + R_2[\text{vers }(O_1 + O_2) - \text{vers } O_1],$$

or $t_2 = (R_1 - R_2) \text{ vers } O_1 + R_2 \text{ vers }(O_1 + O_2);$

$$t_3 = t_2 + P_2d - P_3e = R_1 \text{ vers } O_1 + R_2[\text{vers }(O_1 + O_2) - \text{vers } O_1]$$
$$+ R_3[\text{vers }(O_1 + O_2 + O_3) - \text{vers }(O_1 + O_2)],$$

or $t_3 = (R_1 - R_2) \text{ vers } O_1 + (R_2 - R_3) \text{ vers }(O_1 + O_2)$
$+ R_3 \text{ vers }(O_1 + O_2 + O_3).$

The law connecting the terms of these expressions is very simple, and therefore the expression for *any* tangent may be written at once.

Now $P_1V_1 = \dfrac{t_1}{\sin O_1} = \dfrac{R_1 \text{ vers } O_1}{\sin O_1} = R_1 \tan \tfrac{1}{2}O_1,$

as before found.

$$\left.\begin{aligned} P_2V_2 &= \frac{t_2}{\sin(O_1+O_2)} = \frac{R_1 \operatorname{vers} O_1 + R_2[\operatorname{vers}(O_1+O_2) - \operatorname{vers} O_1]}{\sin(O_1+O_2)}; \\ P_3V_3 &= \frac{\begin{array}{c} R_1 \operatorname{vers} O_1 + R_2[\operatorname{vers}(O_1+O_2) - \operatorname{vers} O_1] \\ + R_3[\operatorname{vers}(O_1+O_2+O_3) - \operatorname{vers}(O_1+O_2)] \end{array}}{\sin(O_1+O_2+O_3)}. \end{aligned}\right\} \quad (a)$$

The tangents P_1V_1 and P_2V_2 are scarcely needed.

Again:

$$AV_1 = R_1 \tan \tfrac{1}{2}O_1.$$

Considering any pair of tangents, AV_2 and P_2V_2 for example, and noting that their relations to the radii and central angles are identical, except that they stand at the opposite ends of the curve, it is plain that the formula for AV_2 may be derived from that of P_2V_2 or *vice versa*, by simply interchanging R_1 and R_2, as well as O_1 and O_2.

For the same reason, the formula for AV_3 may be derived from that of P_3V_3, or *vice versa*, by substituting R_3, R_2, and R_1, in order, in place of R_1, R_2, and R_3, in order, and doing the same with the central angles.

This amounts to interchanging R_1 and R_3 as well as O_1 and O_3.

Thus we have

$$\left.\begin{aligned} AV_2 &= \frac{R_2 \operatorname{vers} O_2 + R_1[\operatorname{vers}(O_1+O_2) - \operatorname{vers} O_2]}{\sin(O_1+O_2)}; \\ AV_3 &= \frac{\begin{array}{c} R_3 \operatorname{vers} O_3 + R_2[\operatorname{vers}(O_2+O_3) - \operatorname{vers} O_3] \\ + R_1[\operatorname{vers}(O_1+O_2+O_3) - \operatorname{vers}(O_2+O_3)] \end{array}}{\sin(O_1+O_2+O_3)}. \end{aligned}\right\} \quad (a')$$

Example.—Let $AO_1P_1 = O_1 = 38°$, $P_1O_2P_2 = O_2 = 18°$, $AO_1 = R_1 = 600$, and $P_2O_2 = R_2 = 900$.

To find AV_2 and P_2V_2.

From the above formula we have

$$P_2V_2 = \frac{600 \times .2120 + 900 \times .2288}{.8290} = \frac{333.1}{.8290} = 401.8,$$

and $$AV_2 = \frac{900 \times .04894 + 600 \times .39187}{.82904} = \frac{279.17}{.82904} = 336.7.$$

If we draw lines parallel and perpendicular to P_3V_3 instead of to AV_3, as in Fig. 102, we may compute AV_2, AV_3, etc., directly, *Problem.*—To determine a simple curve tangent to a compound curve at any point.

Let P_2 (Fig. 102) be the point. Make $AA' = P_2V_2 - AV_2$.

We readily find

$$P_2V_2 - AV_2 = \frac{(R_2 - R_1)[\text{vers}\,(O_1 + O_2) - (\text{vers}\,O_1 + \text{vers}\,O_2)]}{\sin\,(O_1 + O_2)}.$$

This gives the tangent point A'. Then

$$A'E = R = A'V_2 \cot \tfrac{1}{2}(O_1 + O_2).$$

Observe that P_2 represents any point on the arc P_1P_2.

In the same manner the elements of a simple curve may be found tangent to a compound curve at any point.

Problem.—Two curved railway tracks cross each other. To find the angles at the crossings of the rails, and the relations between them. Let O be the center of the track BA and DC, and O_1 the center of the track AC and BD. Draw the radii OA, OB, etc., O_1A, O_1B, etc. OD and O_1D not drawn on the figure.

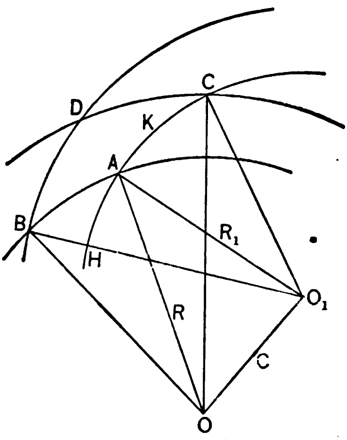


FIG. 103.

Let $AO = R$, $AO_1 = R_1$, $BO_1 = R_1 + g$, $CO = R + g$, and $OO_1 = c$. g is the gauge of the tracks.

The triangle AOO_1 gives

$$\cos A = \frac{R^2 + R_1^2 - c^2}{2RR_1};$$

$$\text{vers}\,A = \frac{(c + R - R_1)(c - R + R_1)}{2RR_1}.$$

Similarly, from the triangles BOO_1, COO_1, and DOO_1, we have

$$\cos B = \frac{R^2 + (R_1 + g)^2 - c^2}{2R(R_1 + g)},$$

$$\text{vers}\,B = \frac{(c + R - R_1 - g)(c - R + R_1 + g)}{2R(R_1 + g)};$$

$$\cos C = \frac{R_1^2 + (R+g)^2 - c^2}{2R_1(R+g)},$$

$$\operatorname{vers} C = \frac{(c + R_1 - R - g)(c - R_1 + R + g)}{2R_1(R+g)};$$

and
$$\cos D = \frac{(R+g)^2 + (R_1+g)^2 - c^2}{2(R+g)(R_1+g)},$$

$$\operatorname{vers} D = \frac{(c + R - R_1)(c - R + R_1)}{2(R+g)(R_1+g)}.$$

These equations give the angles sought.

Any two of these equations will give the *relation* between the two angles involved.

Thus, equating the values of c^2 in the first and third equations above, we have

$$R^2 + (R_1+g)^2 - 2R(R_1+g)\cos B = R^2 + R_1^2 - 2RR_1 \cos A.$$

From this we readily find

$$\cos B = \frac{R_1}{R_1+g}\cos A + \frac{g(R_1 + \frac{1}{2}g)}{R(R_1+g)}. \quad . \quad . \quad . \quad (29)$$

Or, adding $\frac{\frac{1}{2}g^2}{R(R_1+g)}$ to the second member of the equation, we have

$$\cos B = \frac{R_1}{R_1+g}\cos A + \frac{g}{R}, \textit{ very nearly.} \quad . \quad (30)$$

Similarly for any two of the angles.

Writing the approximate values only, which are very nearly correct, we have

$$\cos C = \frac{R}{R+g}\cos A + \frac{g}{R_1}, \text{ very nearly; } . \ . \ . \ . \ . \ . \ . \ . \quad (31)$$

$$\cos D = \frac{RR_1}{(R+g)(R_1+g)}\cos A + \frac{g}{R+g} + \frac{g}{R_1+g}, \text{ very nearly; } (32)$$

$$\cos C = \frac{R(R_1+g)}{R_1(R+g)}\cos B + \frac{g}{R_1} - \frac{g}{R+g}, \text{ very nearly; } . \ . \quad (33)$$

$$\cos D = \frac{R}{R+g}\cos B + \frac{g}{R_1+g}, \text{ very nearly; } . \ . \ . \ . \ . \quad (34)$$

$$\cos D = \frac{R_1}{R_1 + g} \cos C + \frac{g}{R + g}, \text{ very nearly. . . . . . } \quad (35)$$

Let d = the degree of the curve of the rail AB, D that of CD, d_1 that of AC, and D_1 that of BD.

In case the angles OAO_1, OBO_1, exceed 90°. Let A, B, etc., represent their supplements, which are the angles of intersection of the rails. Then

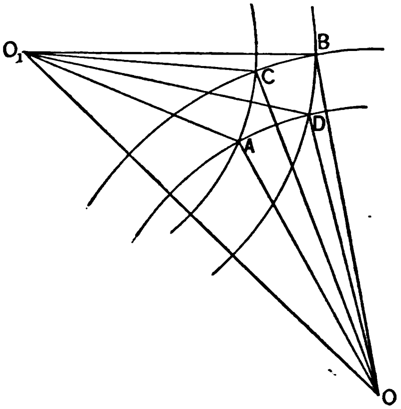


FIG. 104.

$$\cos A = \frac{c^2 - (R^2 + R_1^2)}{2RR_2},$$

$$\text{vers } A = \frac{(R + R_1 + c)(R + R_1 - c)}{2RR_1};$$

$$\cos B = \frac{c^2 - R^2 - (R_1 + g)^2}{2R(R_1 + g)},$$

$$\cos C = \frac{c^2 - R_1^2 - (R + g)^2}{2R_1(R + g)},$$

and

$$\cos D = \frac{c^2 - (R + g)^2 - (R_1 + g)^2}{2(R + g)(R_1 + g)}.$$

Since these values are the same as in the former case, with a negative sign, it is evident that if we eliminate c from them we will obtain equations the same as equations (30) to (35), with the sign of the last or absolute term changed. Thus we have

$$\cos B = \frac{R_1}{R_1 + g} \cos A - \frac{g}{R}, \text{ very nearly; } \ldots \ldots \quad (36)$$

$$\cos C = \frac{R}{R + g} \cos A - \frac{g}{R_1}, \text{ very nearly; } \ldots \ldots \quad (37)$$

$$\cos D = \frac{RR_1}{(R + g)(R_1 + g)} \cos A - \frac{g}{R+g} - \frac{g}{R_1+g}, \text{ very nearly; } \quad (38)$$

$$\cos C = \frac{R(R_1 + g)}{R_1(R + g)} \cos B - \frac{g}{R_1} + \frac{g}{R + g}, \text{ very nearly; } \ldots \quad (39)$$

$$\cos D = \frac{R}{R + g} \cos B - \frac{g}{R_1 + g}, \text{ very nearly; } \ldots \ldots \quad (40)$$

$$\cos D = \frac{R_1}{R_1 + g} \cos C - \frac{g}{R + g}, \text{ very nearly. } \ldots \ldots \quad (41)$$

We may find results practically true by the following short and simple method.

Let L in Fig. 103 mark the intersection of AB with the radius CO. Now the sum of the angles in the triangles AOO_1 and BOO_1 being equal, we have

$$B + BOA + AOO_1 + OO_1B = A + AO_1B + OO_1B + AOO_1.$$

$$\therefore\ B = A + AO_1B - AOB.$$

Now $\quad BH = g,$ and $BAH = A$, nearly.

Let H be the intersection of the arc CA and the radius BO_1.

Then $\quad AH = \dfrac{g}{\tan A}$, nearly, and $AB = \dfrac{g}{\sin A}$, nearly.

The angle $\quad AO_1B = \dfrac{AH \times d_1}{100} = \dfrac{d_1 g}{100 \tan A}$,

and the angle $\quad AOB = \dfrac{AB \times d}{100} = \dfrac{dg}{100 \sin A}$.

Hence $$B = A + \frac{g}{100}\left(\frac{d_1}{\tan A} - \frac{d}{\sin A}\right). \ldots \quad (42)$$

Similarly, $$C = A + \frac{g}{100}\left(\frac{d}{\tan A} - \frac{d_1}{\sin A}\right). \quad . \quad . \quad . \qquad (43)$$

Example.—Let $R = 1228$, $R_1 = 764$. $\therefore\ d = 4° \ 40'$, and $d_1 = 7° \ 30'$. Also, $g = 4.708$, and $A = 20°$, to find B and C.

Eq. (30) gives

$$\cos B = .993875 \times .93969 + .003834 = .937869.$$

$$\therefore\ B = 20° \ 18' +.$$

Or eq. (42) gives

$$B = 20 + .04708(20.606 - 13.644) = 20° + 0°.32777 = 20° \ 19' +.$$

Eq. (31) gives

$$\cos C = .99618 \times .93969 + .006162 = .94226$$

$$\therefore\ C = 19° \ 34'.$$

Or eq. (43) gives

$$\cos C = 20 + .04708(12.82 - 21.93) = 19°.57 = 19° \ 34'.$$

CHAPTER VII.

REVERSED CURVES.

REVERSED curves are mostly used in connection with turnouts where the velocity is slow and space an object. They are especially objectionable on the main line since, adjacent to the reversing point, the outer rails of the two branches are on opposite sides of the track. This calls for the elevation of both sides at once, which is impossible, and so the track must be level at that point and cannot therefore be properly elevated at any point near it.

Theorem.—The reversing point of a reversed curve between parallel tangents is in the line joining the tangent points. Let *AMB* be such a curve reversing at *M*, the radii of the two branches being equal or unequal.

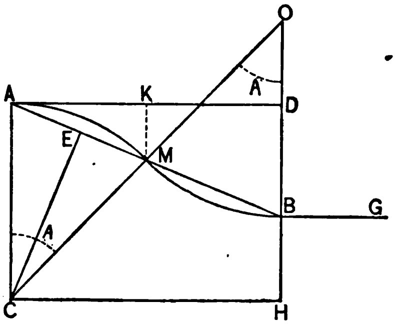


FIG. 105.

Draw the chords *AM* and *BM*; and *CH* parallel to *AD*, to meet *OB* prolonged; also the radii as shown. The radii *CA* and *BO*, being perpendicular to the parallel tangents, are parallel, and the radii CM and MO, being perpendicular to the common tangent at M are in the same straight line. Hence

$$ACM = BOM.$$

$$\therefore \quad 90° - \tfrac{1}{2}ACM = 90° - \tfrac{1}{2}BOM,$$

or

$$AMC = BMO.$$

Hence AM and MB are in the same straight line, that is, M is on the line AB.

I. To connect two parallel tangents by a reversed curve having equal radii.

Given the perpendicular distance between two parallel tangents, $BD = b$ (Fig. 105), and the common radius $CM = MO$, to find $AD = a$ and the chords AM and BM.

Let $AB = c$. Since the radii are equal and the angle $ACM = BOM$, the triangles ACM and BMO are equal, and $AM = BM = \frac{1}{2}c$.

Hence when the radii are equal the chords AM and BM are equal. Draw CE perpendicular to AM.

Then

$$AE = \tfrac{1}{2}AM = \tfrac{1}{4}c,$$

$$\text{angle } BAD = \tfrac{1}{2}ACM = ACE,$$

and hence the right triangles BAD and ACE are similar and give

$$AC \div AE = AB \div BD,$$

or

$$R \div \tfrac{1}{4}c = c \div b,$$

$$\therefore \quad c = 2\sqrt{bR}, \quad \text{or} \quad R = \frac{c^2}{4b}, \quad \text{or } b = \frac{c^2}{4R}. \quad . \quad . \quad (1)$$

But

$$c^2 = a^2 + b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \quad R = \frac{a^2 + b^2}{4b}, \quad \text{or} \quad a = \sqrt{b(4R-b)}, \quad \text{or} \quad b = 2R - \sqrt{4R^2 - a^2}. \quad (3)$$

When any two of the quantities a, b, c, and R are given the other two are readily found by these equations.

Again, $OH = OB + DH - BD = 2R - b,$

and $CO = 2R.$

Let $ACM = BOM = A.$

Then $$\cos A = \frac{2R - b}{2R}, \quad \text{or} \quad \text{vers}\,A = \frac{b}{2R}. \quad . \quad . \quad . \quad (4)$$

$$\left. \begin{array}{l} AD = CH = 2R \sin A, \quad AM = 2AE = 2R \sin \tfrac{1}{2}A, \\ \text{and } AB = 2AM = 4R \sin \tfrac{1}{2}A. \end{array} \right\} \quad . \quad (5)$$

Example.—Let $b = 20$, and $R = 741.7$. $\therefore\ D = 7°.725$. To find AD, AM, etc.

$$\text{vers}\,A = \frac{20}{1483.4} = .01348. \quad \therefore\ A = 9°\ 25' = 9°.417.$$

Then $$AD = 2R \sin A = 1483.4 \times .163613 = 242.70;$$

$$AM = 2R \sin \tfrac{1}{2}A = 1483.4 \times .08209 = 121.77.$$

Also, $$\text{arc } AM = 100\frac{9.417}{7.725} = 121.90.$$

If AD and BD are given, then

$$\tan \tfrac{1}{2}A = \frac{BD}{AD}; \quad AB = \frac{BD}{\sin \tfrac{1}{2}A}; \quad \text{and} \quad R = \frac{AE}{\sin \tfrac{1}{2}A} = \frac{AB}{4 \sin \tfrac{1}{2}A}.$$

Example.—Let $BD = 13$, and $AD = 200$. Then

$$\tan \tfrac{1}{2}A = \frac{13}{200} = .065.$$

$$\therefore\ A = 3°\ 43'\ 08''; \quad \sin \tfrac{1}{2}A = .06486;$$

$$AB = \frac{13}{.06486} = 200.43;$$

$$R = \frac{200.43}{.25944} = 772.55.$$

II. To connect two parallel tangents by a reversed curve having unequal radii. (Fig. 106.)

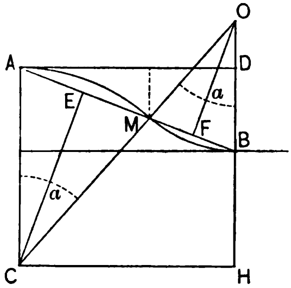


FIG. 106.

We have

$$CO = CM + MO = R + r,$$

and

$$HO = OB + AC - BD = R + r - b.$$

$$\therefore \cos A = \frac{R + r - b}{R + r},$$

or $$\operatorname{vers} A = \frac{b}{R + r}.$$

Then

$$AD = CH = (R + r) \sin A;$$

$$AM = 2R \sin \tfrac{1}{2}A; \quad BM = 2r \sin \tfrac{1}{2}A;$$

$$AB = AM + BM = 2(R + r) \sin \tfrac{1}{2}A.$$

Again, let $AM = c'$, $BM = c''$, $AB = c' + c'' = c$. Then, as above, from triangles ACE and ABD,

$$R = \frac{c'c}{2b},$$

and, from triangles BFO and ABD,

$$r = \frac{c''c}{2b}.$$

Adding gives

$$R + r = \frac{c^2}{2b} = \frac{a^2 + b^2}{2b}.$$

This shows that for the same distances AD and BD the *sum of the radii is constant.*

1. Given the perpendicular distance BD between tangents, the radii $CM = R$ and $MO = r$, to find AD, the chords AM and MB, and the central angles A.

Example.—Let $BD = 28$, $R = 955$, $r = 574$.

$$\therefore \text{ vers } A = \tfrac{28}{1529} = .01831. \quad \therefore A = 10° \, 59'.$$

$$\sin A = .19052, \quad \text{and} \quad \sin \tfrac{1}{2}A = .09570.$$

$$AD = 1529 \times .19052 = 291.50;$$

$$AM = 1910 \times .09570 = 182.79;$$

$$BM = 1148 \times .09570 = 109.86.$$

2. Let AD, BD, and R be given, to find the other radius, r, the chords, and the central angles.

Example.—Let $AD = 400$, $BD = 56$, and $R = 900$. We have

$$\tan \tfrac{1}{2}A = \tfrac{56}{400} = .14. \quad \therefore \tfrac{1}{2}A = 7° \, 58'.$$

$$AB = BD \div \sin \tfrac{1}{2}A = 56 \div .13802 = 404.04;$$

$$AM = 2\,R \sin \tfrac{1}{2}A = 1800 \times .13860 = 249.48;$$

and

$$BM = AB - AM = 154.56.$$

$$r = \tfrac{1}{2}BM \div \sin \tfrac{1}{2}\,A = 77.28 \div .13860 = 557.6.$$

III. To unite two tangents not parallel by a reversed curve of common radius. (Fig. 107.)

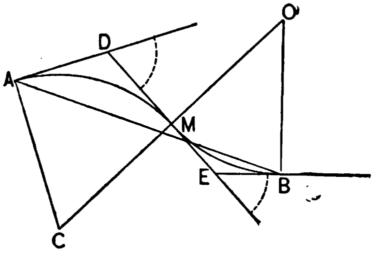


FIG. 107.

Given the angles of intersection at D and E, and the length of the common tangent DE, to find the common radius $CM = MO = R$ to unite the tangents AD and BE.

We have

$$DM = R \tan \tfrac{1}{2}D;$$

$$EM = R \tan \tfrac{1}{2}E.$$

$$\therefore DE = R(\tan \tfrac{1}{2}D + \tan \tfrac{1}{2}E),$$

or

$$R = \frac{DE}{\tan \tfrac{1}{2}D + \tan \tfrac{1}{2}E}.$$

Example.—Let $DE = 250$, $D = ACM = 24°$, $E = BOM = 16°$.

$$\tan 12° + \tan 8° = .35310.$$

$$\therefore R = 250 \div .35310 = 708.02.$$

IV. To connect two given points on two diverging tangents by a reversed curve of common radius. (Fig. 108.)

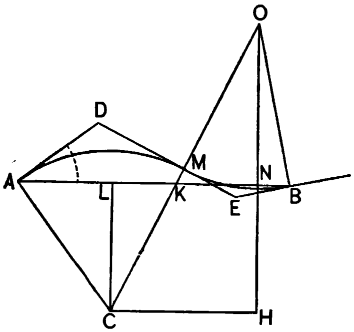


Fig. 108.

Given the angles $BAD = A$, $ABE = B$, also AB a, to find the radius R of the curve AMB.

Draw CL and ONH perpendicular, and CH parallel, to AB.

Let

$$HCO = BKO = CKA = K.$$

Then

$$HO = CO \sin K = 2R \sin K.$$

Also, $\quad HO = CL + NO = R \cos A + R \cos B.$

$\therefore \quad 2R \sin K = R \cos A + R \cos B,$

or $\quad \sin K = \dfrac{\cos A + \cos B}{2}.$

Again, $AL = R \sin A$; $LN = CH = CO \cos K = 2R \cos K$;
$Bn = R \sin B.$

Adding gives

$$a = R(\sin A + \sin B + 2 \cos K),$$

or $\quad R = \dfrac{a}{\sin A + \sin B + 2 \cos K}$

Also, $\quad ACK = ACL + LCK = A + 90° - K,$
and $\quad BOM = B + 90° - K.$

The foregoing solution requiring barely a single division, except by the factor 2, may be compared with Henck Problem 40.

Example.—Let $a = 1500$, $A = 18°$, $B = 6°$, to find R.

$\cos A = .95106$
$\cos B = .99452 \qquad \sin A = .30902$
$2)1.94558 \qquad \sin B = .10453$
$\sin K = .97279 \qquad \therefore \; 2 \cos K = .46340$

Then $R = 1500 \div .87695 = 1710.47.$

V. To connect two diverging tangents by a reversed curve, starting at a given point.

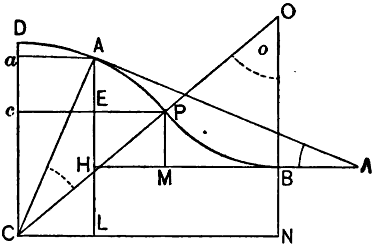


FIG. 109

(*a*) Advancing toward the intersection of tangents. (Fig. 109.) Given the angle of intersection $AKB = CAL = K$, the initial point A, the distance $AK = a$, and the radii $CP = R$ and $OP = r$, to find the central angles C and O of the reversed curve APB.

First Solution.—We have

$$AL = R \cos K; \quad AH = AK \sin K;$$

and $$BN = AL - AH = R \cos K - a \sin K.$$

Then

$$\cos O = \frac{NO}{CO} = \frac{BN + r}{R + r}, \quad \text{and} \quad C = DCP - ACD = O - K.$$

Now having the central angles C and O, run the branch AP subtending the angle C; then reverse and run the branch PB subtending the angle O.

Example.—Let $K = 24° 30'$, $AK = 854$, $R = 1440$, and $r = 1146$

Then $AL = 1440 \times .91 = 1310.4$

$AH = 854 \times .4147 = 354.1$

$BN = 956.3$

$r = 1146$

$(Bn + r = 2102.3) \div 2586 = .81291.$

$\therefore \quad O = 35° 37'$, and $C = O - K = 11° 07'$.

If convenient to measure AH instead of AK, the solution is shortened.

Second Solution.—Suppose the curve run backward to D where the tangent is parallel to BK, and draw lines as shown. Then

$$Dc = R \text{ vers } O; \quad PM = r \text{ vers } O.$$

$$\therefore \quad PM = Dc\frac{r}{R} = Da\frac{r}{R} + AE\frac{r}{R}.$$

Hence $\quad AE + PM = AH = b = AE\left(\frac{r+R}{R}\right) + Da\frac{r}{R}.$

But $\quad Da = R \text{ vers } ACD = R \text{ vers } K.$

$$\therefore \quad AE\left(\frac{r+R}{R}\right) + r \text{ vers } K = b, \text{ or } AE = (b - r \text{ vers } K)\frac{R}{R+r}.$$

If $r = R$, $\quad AE = \frac{b - R \text{ vers } K}{2}.$

Now measure AE as found, making angle $CAL = K$. Run EP perpendicular to AE, and the curve AP to intersect EP in P. Reverse and run the curve PB. This method does not require the central angles to be known, and neither method requires the curve to be actually run backward to D. The problem is thus much shortened.

(*b*) Receding from the intersection of tangents. (Fig. 110.)

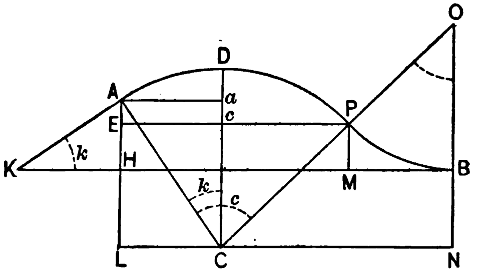


FIG. 110.

Given the angle of intersection $AKB = CAL = K$, the distance $AK = a$, and the radii $CA = R$ and $OB = r$, to find the central angles C and O of the reversed curve APB.

Draw Aa parallel to BK. Then

$$Da = R \text{ vers } K; \quad AH = AK \sin K;$$

$$Da + AH = DE = b;$$

$$ON = OB + DC - ED = R + r - b.$$

$$\therefore \quad \cos O = \frac{R + r - b}{R + r}, \quad \text{and} \quad \text{vers } O = \frac{b}{R + r}.$$

Evidently, $\qquad C = K + O.$

The solution here given is, of course, applicable to the former case (Fig. 109); and the first solution there given is likewise applicable here.

Example.—Let $K = 18° 23'$, $AK = 920$, $R = 955$, and $r = 819$.

$$\begin{aligned} Da &= R \text{ vers } K = 955 \times .05103 = 48.73 \\ AH &= AK \sin K = 920 \times .31537 = \underline{290.14} \\ &\qquad\qquad\qquad\qquad\qquad\qquad 338.87 \end{aligned}$$

$$338.87 \div 1774 = .19101.$$

$$\begin{aligned} \therefore \quad O &= 36° \, 00' \\ K &= \underline{18° \, 23'} \\ C &= 54° \, 23' \end{aligned}$$

Second Solution.

$$Dc = R \text{ vers } DCP = R \text{ vers } O; \quad PM = r \text{ vers } O.$$

$$\therefore \; PM = Dc\frac{r}{R} = Da\frac{r}{R} + AE\frac{r}{R}.$$

Hence $\qquad AE + PM = AH = b = AE\left(\frac{R + r}{R}\right) + Da\frac{r}{R}.$

But $\qquad Da = R \text{ vers } ACD = R \text{ vers } K.$

$$\therefore \; AE\left(\frac{R + r}{R}\right) + r \text{ vers } K = b, \quad \text{or} \quad AE = (b - r \text{ vers } K)\frac{R}{R + r}.$$

If $r = R$, $\qquad AE = \frac{b - R \text{ vers } K}{2}.$

Now run AE and EP to meet the curve ADP, etc., as in a former case.

VI. To connect two given points on two diverging tangents by a reversed curve.

Given the chord $AB = a$, the angles $BAL = A$ and $ABK = B$, and the radius $AC = R$, to find the radius $BO = r$ of the reversed curve APB. (Fig. 111.)

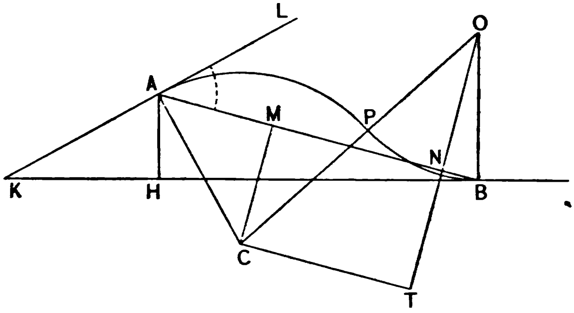


FIG. 111.

Draw CT parallel and CM and ONT perpendicular to AB.

$$CT = MN = AB - AM - BN = a - R \sin A - r \sin B,$$

and $$OT = CM + ON = R \cos A + r \cos B.$$

Also, $$CO = R + r.$$

$$\therefore (R + r)^2 = (a - R \sin A - r \sin B)^2 + (R \cos A + r \cos B)^2.$$

Expanding and observing that

$$1 - (\sin A \sin B + \cos A \cos B) = 1 - \cos(A - B) = \text{vers}(A - B),$$

we readily find

$$r = \frac{\frac{1}{2}a - R \sin A}{\frac{R}{a}\text{vers}(A - B) + \sin B}$$

Compare with Henck Problem 38.

Example.—Let $a = 1500$, $R = 1600$, $A = 18°$, and $B = 6°$, to find r. Then

$$\tfrac{1}{2}a = 750$$

$$R \sin A = \underline{494.43}$$

$$255.57$$

$$\frac{R}{a} \operatorname{vers} (A - B) = \frac{16}{15} .02185 = .02331$$

$$\sin B = \underline{.10453}$$

$$.12784$$

$$r = 255.57 \div .12784 = 1999.14.$$

VII. Having located a reversed curve, it is required to shift the P.R.C. so that the terminal branch may end in a tangent parallel to the original tangent. (Fig. 112.)

Let APB be the curve, $CP = R$, $OP = r$, and $BK = D$, the distance between the original tangent BL and the new tangent $B'K$, to find the angle of retreat PCP'.

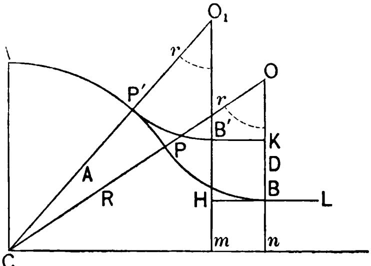


FIG. 112.

Draw Cmn parallel to BL. We have

$$O_1m = O_1B' + B'H + Hm = r + D + Hm,$$

and $$On = OB + Hm = r + Hm;$$

$$\therefore\ O_1m - On = D.$$

That is, $$(R + r) \cos O_1 - (R + r) \cos O = D;$$

$$\therefore\ \cos O_1 = \cos O + \frac{D}{R + r}.$$

Then $$OCO_1 = A = O_1Cn - OCn$$

$$= (90° - O_1) - (90° - O) = O - O_1.$$

Example.—Let $O = 34° \, 20'$, $D = 94$, $R = 1433$, and $r = 879$,

We have

$$\cos O = .82577$$

$$D \div (R + r) = \underline{.04174}$$

$$\therefore\ \cos O_1 = .86751$$

$$\therefore\ O_1 = 29^\circ\ 50';$$

$$\therefore\ A = O - O_1 = 4^\circ\ 30'.$$

CHAPTER VIII.

TURNOUTS.

A **TURNOUT** is a curved track leading out from a main track. At the place where the outer rail or "lead" crosses the rail of the main track a frog is placed, which allows the flanges of the car-wheels to pass the rails.

A frog is a piece of metal, usually steel, having two straight channels crossing each other on the upper surface through which the flanges of the wheels pass. The triangular part of the upper surface formed by the channels is called the tongue of the frog (Fig. 113). Every railway company should own a variety of frogs suited to different turnouts that are likely to be required.

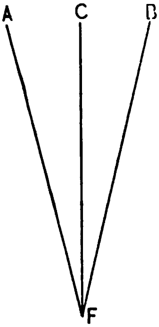


FIG. 113.

Frogs are usually designated by numbers, the number of a frog being the number of times the bisecting line CF contains the base line AB. Denoting the number by n, we have $n = \frac{CF}{AB}$, and $2n = \frac{CF}{AC}$. Designating the frog angle AFB by F, we have

$$\cot \tfrac{1}{2}F = \frac{CF}{AC}.$$

$$\therefore \quad \cot \tfrac{1}{2}F = 2n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Standard frogs are made so that n is a whole number or a whole number plus $\frac{1}{2}$, and therefore $2n$ is a whole number.

Let Fig. 114 represent a "split-switch" turnout. AB and HF represent the rails of the main track, and AC one of the switch-rails. In the figure the switch is set for the turnout, but when set for the main track A takes the position of A'.

Prolong the curve backward to c, where the radius cLO is perpendicular to the main track.

Draw also the radii CO and FO; Ca parallel to the main track and the tangents CVK and FV.

Then $LFV = LOF =$ the frog angle, which we will denote by F. Let $S =$ the switch angle $BAC = FKV = LOC$, then

$$COF = FVK = F - S.$$

Then $$CFV = \tfrac{1}{2}FVK = \tfrac{1}{2}(F - S).$$

Also, $DFC = LFV - CFV = F - \frac{1}{2}(F - S) = \frac{1}{2}(F + S)$.

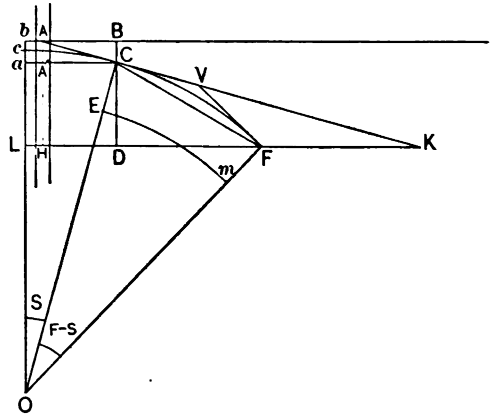


FIG. 114.

Or, since the tangent CV makes an angle S, and the tangent FV an angle F with the main track,

$$\text{angle } FVK = COF = F - S.$$

Hence $$CFV = \tfrac{1}{2}(F - S),$$

and $$DFC = DFV - CFV = F - \tfrac{1}{2}(F - S) = \tfrac{1}{2}(F + S).$$

Let $R' = R + \frac{1}{2}g = CO =$ the radius of the outer or "lead" rail, and $D' =$ the degree of the curve CF.

Let $d = AA' = BC$, which is properly the throw of the switch, and $l =$ the length of the switch-rail.

Suppose $d = 5$ inches $= .416$ of a foot, and $l = 15$ feet.

Then $$\sin S = .\frac{416}{15} = .027.$$

$$\therefore\ S = 1^\circ\ 35'\ 30'', \quad \text{and} \quad \text{vers}\ S = .0003858.$$

Now $\qquad cL = R'\ \text{vers}\ F,\ \text{and}\ \ ca = R'\ \text{vers}\ S.$

$$\therefore\ R'\ \text{vers}\ F - R'\ \text{vers}\ S = cL - ca = aL = CD = g - d.$$

$$\therefore\ R' = \frac{g - d}{\text{vers}\ F - \text{vers}\ S} = \frac{g - d}{\cos S - \cos F},$$

or $\qquad \text{vers}\ F = \text{vers}\ S + \dfrac{g - d}{R'}. \quad \dots\dots\dots \quad (2)$

We observe that

$$\text{vers}\ F - \text{vers}\ S = (1 - \cos F) - (1 - \cos S) = \cos S - \cos F,$$

so that in all such cases either the cosines or the versines may be used as is most convenient.

Again, $\qquad CF = \dfrac{CD}{\sin CFD} = \dfrac{g - d}{\sin \frac{1}{2}(F + S)}, \quad \dots\dots \quad (3)$

$$DF = CD \cot CFD = (g - d) \cot \tfrac{1}{2}(F + S), \quad . \quad (4)$$

$$HF = l + DF = l + (g - d) \cot \tfrac{1}{2}(F + S), \quad . \quad (5)$$

and $\qquad LF = FO \sin LOF = R' \sin F. \quad \dots\dots \quad (6)$

HF is the "lead" and shows the distance apart, measured parallel to the main track, of the head-block and the *point* of the frog.

Since D' represents the degree of the curve CF, we have

$$\text{arc}\ CF = 100\ \frac{F - S}{D'}.$$

The lead can be lengthened or shortened somewhat without materially damaging the turnout curve; and therefore, in practice, rails of such lengths are selected, when possible, as will make the lead sufficiently near the true length, thus avoiding the cutting of rails.

In case of sharp turnouts (those of short radii), point or switch rails shorter than 15 feet should be used; but in any case to find the radius or the lead, etc., it is only necessary to substitute in the above equations the proper values of d and S.

Table VI gives the elements of turnouts for a variety of frogs, for switch-rails of 15 feet in length, and for a throw of 5 inches;

and also for a number of frogs for the same throw and for switch-rails 8 feet in length.

The same table gives also, in case of a double turnout on opposite sides of a straight track, the number (n''), angle (F''), and position of the middle frog, corresponding to given turnout frogs F and F', as shown in Fig. 122. The similar triangles ABC and aOC of Fig. 114 give

$$CO = aC\frac{AC}{BC}, \quad \text{or} \quad R = aC\frac{l}{d}.$$

If in this we put $aC = l$, we have

$$R = \frac{l^2}{d}.$$

This shows the value of R such that the end c of the prolonged curve Cc will be opposite the head-block A.

If $l = 15$ feet and $d = 5$ inches, we have $R = 540$. And if $l = 8$ feet and $d = 5$ inches, $R = 153.6$. Hence, in Table VI, for switch-rail 15 feet

$$LF \gtrless HF \text{ accordingly as } R \gtrless 540,$$

and for switch-rail 8 feet

$$LF \gtrless HF \text{ accordingly as } R \gtrless 153.6.$$

Example.—Find the lead and radius for a number 9 frog, switch-rail 15 feet and throw 5 inches. We have

$$F = 6^\circ\ 21'\ 35'', \quad \text{and} \quad S = 1^\circ\ 35'\ 30''.$$

$$\therefore \quad \text{vers}\ F - \text{vers}\ S = .005768, \quad \text{and} \quad g - d = 4.2917.$$

$$\therefore \quad R' = \frac{4.2917}{.005768} = 744.02. \quad \text{Then } R = 741.67,$$

and

$$HF = l + (g - d) \cot \tfrac{1}{2}(F + S) = 15 + 4.2917 \times 14.3885 = 76.75.$$

A turnout curve between head-block and frog is in general not the same as the main part of the curve beyond the frog. When, therefore, two intersecting tracks or a straight line and track, for example, are to be connected by a compound curve, the tangent points must be found so that with given frogs the connecting curve may be properly located.

Problem.—To connect a straight track and straight line by a compound curve of two branches, the first branch being a turnout curve fitting a given frog. Eqs. (1) to (5) of compound curves apply to this case without change.

Let AV represent the center line of the track, BV the straight line, and APB the center line of the compound curve.

Example.—Let $AO_1 = R_1 = 437$ to fit a No. 7 frog, $R_2 = BO_2 = 537$, $V = 50°$, and $O_1 = 10°$, which is sufficient to clear the frog. It is required to find the tangent point A.

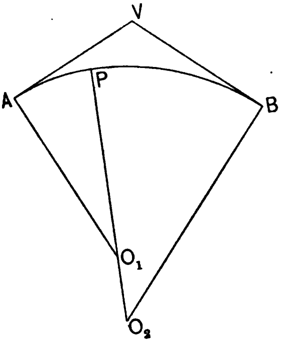


FIG. 115.

Eq. (4) gives

$$AV = T_1 = \frac{R_1 \text{ vers } V + (R_2 - R_1) \text{ vers } O_2}{\sin V}$$

$$= \frac{437 \times .357 + 100 \times .234}{.766} = 234.2.$$

The tangent VB is not generally needed, but may be found in the same way from eq. (5).

Problem.—To connect two intersecting straight tracks by a compound curve of three branches, the two turnout curves having radii to suit given frogs.

Let Fig. 116 represent the case, AP_1 and BP_2 being the center lines of the turnout curves. The tangents are required.

First Method.—The equation under Fig. 102, compound curves, applies directly, and we have

$$\left.\begin{aligned} BV \sin V &= (R_1 - R_2) \text{ vers } O_1 + (R_2 - R_3) \text{ vers } (O_1 + O_2) \\ &\qquad + R_3 \text{ vers } (O_1 + O_2 + O_3), \\ \text{or, } BV \sin V &= R_1 \text{ vers } O_1 + R_2[\text{vers } (O_1 + O_2) - \text{vers} O_1] \\ &\qquad + R_3[\text{vers}(O_1 + O_2 + O_3) - \text{vers}(O_1 + O_2)] \end{aligned}\right\}. \qquad (a)$$

To find AV we may interchange R_1 and R_3, also O_1 and O_3 in the above formulas, as explained under Fig. 102, p. 177. Thus, observing that $O_1 + O_2 + O_3 = V$, we have

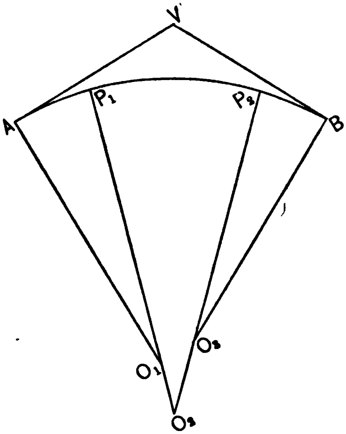


FIG. 116.

$$\left.\begin{aligned}
AV \sin V &= (R_3 - R_2) \operatorname{vers} O_3 + (R_2 - R_1) \operatorname{vers} (O_2 + O_3) \\
&\qquad + R_1 \operatorname{vers} V, \\
\text{and} \\
AV \sin V &= R_3 \operatorname{vers} O_3 + R_2[\operatorname{vers} (O_2 + O_3) - \operatorname{vers} O_3] \\
&\qquad + R_1[\operatorname{vers} V - \operatorname{vers} (O_2 + O_3)].
\end{aligned}\right\} \quad (a')$$

Example 1.—Let $AO_1P_1 = 14° \, 30'$, $P_1O_2P_2 = 31°$, and $P_2O_3B = 14° \, 30'$. Also $AO_1 = 437$ (to fit a number 7 frog), $P_1O_2 = 490$, and $P_2O_3 = 400$. To find AV. Substituting in first of the above equations, we have

$$AV \sin V = -90 \times .03185 + 53 \times .29909 + 437 \times .5$$

$$= -2.87 + 15.85 + 218.5 = 231.48;$$

$$\therefore AV = 231.48 \div .886 = 266.15.$$

Example 2.—Let the values be the same as in example 1, except R_1, which we will suppose equal to $R_3 = 400$. Then

$$R_1 - R_2 = -90, \quad \text{and} \quad R_2 - R_3 = 90.$$

Therefore

$$\begin{aligned} AV \sin V &= 90(.29909 - .03185) + 400 \times .500 \\ &= 90 \times .26724 + 200 = 224.05. \end{aligned}$$

Therefore

$$AV = 224.05 \div .866 = 258.72.$$

Eqs. (*a*) apply without change whatever may be the order of the radii, but the first form, which usually involves the least labor, may contain some negative terms which must be subtracted from the sum of the positive terms.

Second Method.—Let $A'P_1P_2B'$ (Fig. 117) represent a symmetrical compound curve. Prolong the middle arc P_1P_2 to A and B as shown. We will illustrate the method by the following

Example.—Let the vertex angle $= V = 98°$, $A'O_1P_1 = B'O_1P_2 = 9° \ 50'$, which is sufficient to clear the frogs.

Let $A'O_1 = B'O_1 = R_1 = 437.06$ corresponding to number 7 frogs, and $P_1O_2 = R_2 = 358.1$. To find the tangents VA' and VB'.

We have, from eqs. (7), (8), etc., of compound curves,

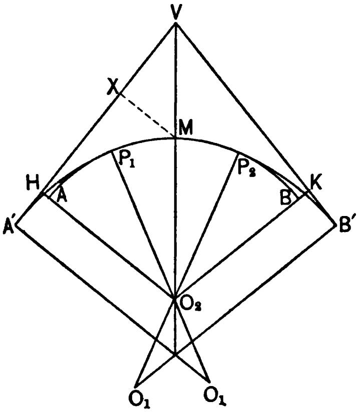


FIG. 117.

$$AH = (R_1 - R_2) \operatorname{vers} O_1 = 79 \times .01469 = 1.16 = a;$$

$$A'H = (R_1 - R_2) \sin O_1 = 79 \times .17078 = 13.49 = d.$$

$\therefore HO_2 = R = 358.1 + 1.16 = 359.26$ = radius of simple curve joining H and K, but not used except to find VH and VA'.

Now $VH = R \tan \frac{1}{2} V = 413.29$, and $VA' = 426.78$.

This gives the tangent point A' (or B') from which the compound curve may be located. The simple curve joining H and K is not shown.

Let Fig. 118 represent the case in which the curve is sharpened at the ends.

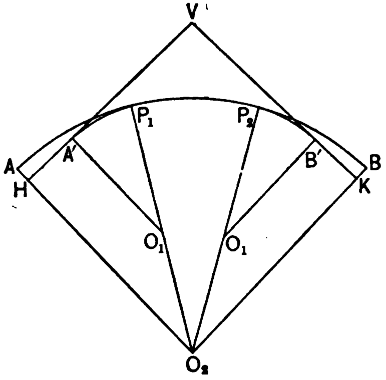


FIG. 118.

Let $V = 84°$, $A'O_1P_1 = B'O_1P_2 = 29°$, $A'O_1 = R_1 = 437.06$, and $P_1O_2 = R_2 = 877.06$.

Then $AH = (R_2 - R_1) \text{ vers } O_1 = 440 \times .12538 = 55.16$;

$A'H = (R_2 - R_1) \sin O_1 = 440 \times .48481 = 213.32$.

$\therefore HO_2 = R = 877.06 - 55.16 = 821.9$.

Now $VH = R \tan \frac{1}{2} V = 740.04$.

$\therefore VA' = 740.04 - 213.32 = 526.72$.

The curve $A'P_1P_2B'$ may now be located.

The simple curve connecting the tangent points H and K is not drawn.

Third Method.—We may easily fit a given turnout curve to a different curve beyond the frog by a simple but direct application of Fig. 81 and corresponding formulas of compound curves.

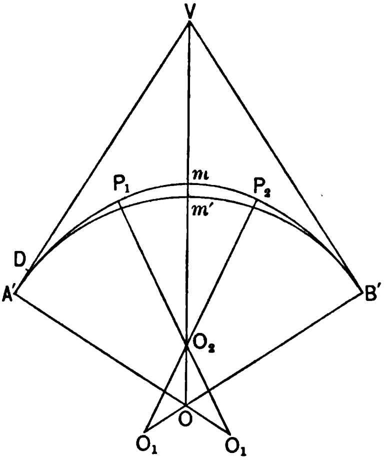


FIG. 119.

Thus let $A'm'B'$, Fig. 119, be a simple curve of radius $A'O = R$ connecting the center lines, $A'V$ and $B'V$, of two tracks; to substitute for it a compound curve $A'mB'$ of two different radii, $A'O_1 = B'O_1 = R_1$ and $P_1O_2 = R_2$, the longer of which is to be used for a distance adjacent to each tangent, for the purpose of fitting a given frog. The tangents and the elements of the curves $A'P_1$ and $B'P_2$ are given.

Example.—Let $V = 98°$, and $A'V = B'V = 4\ 6.78$.

$$\text{Then } R = A'O = BO' = A'V \cot \tfrac{1}{2}V = 426.78 \times .8693 = 371.$$

$$\therefore D = 15° \ 26' \ 40''.$$

Let $A'P_1$ and $B'P_2$ be 13° 06′ 34″ curves for number 7 frogs.

Then $A'O_1 = B'O_1 = R_1 = 437.06$, and $R_1 - R = 66.06$.

Suppose $A'P_1 = B'P_2 = 75$ feet, then

$$O_1 = \tfrac{3}{4}(13° \ 06' \ 34'') = 9° \ 50'.$$

Eq. (17) p. 142 gives

$$R_1 - R_2 = (R_1 - R)\frac{\sin \frac{1}{2}V}{\sin(\frac{1}{2}V - O_1)} = 66.06\,\frac{.75471}{.63158}$$

$$= 66.06 \times 1.195 = 78.94.$$

$$\therefore R_2 = 437.06 - 78.94 = 358.12. \quad \therefore D_2 = 16°.$$

Observe that this curve, $A'P_1P_2B'$, is the same as $A'P_1P_2B'$ of Fig. 117 or of Fig. 118.

We observe that the degrees of the two branches of the compound curve (taking $A'P_1$ and $B'P_2$ together as one) differ approximately from the degree of the simple curve inversely as their lengths.

It is, of course, not necessary to locate any simple curve, but simply to make use in the formulas of the radius ($A'O$, for example) of a simple curve corresponding to any desired tangent $A'V$.

If we suppose the switch-rail to be a part of the turnout *curve*, such curve being tangent to the main track at A (Fig. 120), we have

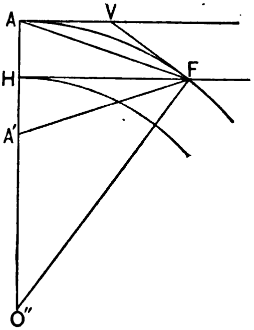


FIG. 120

$$AO'' = \frac{AH}{\text{vers } AO''F};$$

and representing $AO'' = FO''$ by R'' to distinguish it from the true radius $R' = OF$ (Fig. 114), given by eq. (2), we have

$$R'' = \frac{g}{\text{vers } F}. \quad . \quad . \quad . \quad (7)$$

The triangle FHO'' gives

$$HF = R'' \sin F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The triangle AFH gives

$$HF = g \cot \tfrac{1}{2}F, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8')$$

and

$$AF = \frac{g}{\sin \frac{1}{2}F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These equations may be deduced at once from those for Fig. 114 by putting $d = 0$, $S = 0$, and $l = 0$, as they are in this case.

Introducing the number of the frog, we have

$$AFH = \tfrac{1}{2}AO''F = \tfrac{1}{2}HFV = \tfrac{1}{2}F,$$

and

$$HF = AH \cot AFH = g \cot \tfrac{1}{2}F = 2gn. \quad . \quad . \quad . \quad (8'')$$

Or, make $HA' = HA$; then

$$AFA' = 2AFH = F,$$

and the triangle AFA' may represent the frog, and we have, by definition,

$$n = \frac{HF}{AA'} = \frac{HF}{2g}, \quad \text{or} \quad HF = 2gn. \quad . \quad . \quad . \quad (8'')$$

Again,

$$\overline{HF}^2 = AH(2AO'' - AH) = 2AH(AO'' - \tfrac{1}{2}AH) = 2gR_1. \quad (8''')$$

Equating values of HF^2 from (8″) and (8‴), we find

$$\text{the radius of the center line of turnout} = R_1 = 2gn^2. \quad . \quad (10)$$

$$AF^2 = HF^2 + g^2 = 4g^2n^2 + g^2; \quad \therefore \quad AF = g\sqrt{4n^2 + 1}.$$

The foregoing equations are not applicable, however, to straight switch-rails, as we will explain.

Let AC_1F (Fig. 121) represent the turnout curve actually located under the supposition that the switch-rail is a part of the turnout curve, which is tangent to the rail of the main track at the head-block A, AC being the switch-rail. Eq. (10) gives, for a number 9 frog,

$$R = 2 \times 4.708 \times 81 = 762.7.$$

The tangent offset of the curve AC_1F from the rail AV, at a point opposite C, is

$$t = \frac{\overline{15}^2}{2 \times 762.7} = .1475 \text{ feet}$$
$$= 1.77 \text{ inches.}$$

Hence the curve AC_1F is $5 - 1.77 = 3.23$ inches from C, the end of switch-rail. But the real turnout curve must connect C and F and be tan-

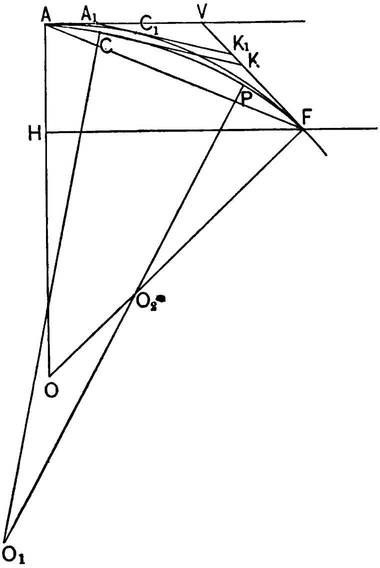


FIG. 121.

gent to the switch-rail and frog. Prolong AC to meet the tangent FK in K.

Now $HF = 2gn$; angle $FAK = \frac{1}{2}F - S$; $AFK = \frac{1}{2}F$.

Also, $\sin AKF = \sin (FAK + AFK) = \sin (F - S)$.

$$\therefore\ AK = AF \frac{\sin \frac{1}{2}F}{\sin (F - S)}, \quad \text{and} \quad FK = AF \frac{\sin (\frac{1}{2}F - S)}{\sin (F - S)}.$$

For a number 9 frog, switch-rail 15 feet and throw 5 inches, we have

$$AK = 56.64, \quad \text{or} \quad CK = 41.64 = T_2, \quad \text{and} \quad FK = 28.30 = T.$$

Since the tangents are unequal a simple curve cannot meet the required conditions, and we will suppose a compound curve of two branches. Since, moreover, the tangents are quite unequal and the central angles small, we know from Chapter VI, eq. (3), and eqs. (*e*) and (*f*) following, that the radii will differ very much. We will suppose the central angles equal, for this will make the radii the most nearly equal possible, as pointed out in Chapter VI, and will result in the best possible curve. The vertex angle

$$CKV = V' = F - S = 6° 21' - 1° 35' = 4° 46',$$

and therefore $\quad O_1 = O_2 = 2° 23'$.

Eqs. (1) and (2), Chapter VI, give

$$CO_1 = R_1 = 1164.13, \quad \text{and} \quad PO_2 = R_2 = 513.85.$$

We note that the offset from the tangent AV to the curve AC_1F is 5 inches at about 25 feet from A, and that the curve has the direction of the switch-rail at about 21 feet from A. By moving the switch-rail forward from AC to A_1C_1 about 8 feet, making AC about 23 feet, a close approximation to the part of the curve C_1F could be used. This would make the lead nearly equal to the true lead given in Table VI. This would also involve some compromise, and would be somewhat troublesome in practice.

Thus is proved the impossibility of locating a simple curve tangent to a switch-rail and to a frog placed in the position determined by the above formulas.

When so placed the result is a turnout difficult and even dangerous to operate, unless the trackman is able to put in by the eye a compound curve answering more or less well the purpose of the turnout.

To Lay Out the Turnout. (Fig. 114.)

When it can be done, which is usually the case, first locate the frog with reference to a joint in the main track. Knowing the length from point of frog to either end of it, we thus find and mark the position of *point* of frog, which is F in Fig. 114.

Measure off FH, giving place of head-block so as to avoid cutting rails if possible.

For example, a number 9 frog calls for a lead $HF = 76.75$. Suppose the length of frog from its point to the switch end, which is the part that applies on the "lead," is 3.35 feet, the switch-rail being 15 feet. This will use up $3.35 + 15.0 = 18.35$ feet, and we will need $76.75 - 18.35 = 58.40$ feet of rails. We may well use two 30-foot rails, making $FH = 3.35 + 15 + 60 = 78.35$. The arc CF is a trifle longer than DF (about 0.1 for a No. 12 and 0.2 for a No. 7 frog), and this will practically allow for expansion and therefore no allowance need be made in measuring FH. Note that $FD = FH - l$.

Remembering that the curve CF is tangent to the switch-rail at C, the curve of the outer rail CF may be located by any methods heretofore explained.

Drawing CF, the curve CF is easily located by ordinates, noting that $CF = DF$ nearly, and that radius $CO = R' = R + \frac{1}{2}g$.

The turnout can be conveniently located by setting the instrument at C, turning off from the line of the track the angle S, which gives the tangent CV, and locating the curve of the outer rail CF. The degree D' of the curve CF, corresponding to $R' = R + \frac{1}{2}g$, may be taken from Table I.

Or set at F and turn off from FD the angle F, which gives the tangent FV, and run the curve both ways from F. This is the handiest and best method.

We may locate the center line as follows:

Make CE perpendicular to AC, equal to $\frac{1}{2}g$, then run the curve from E, with radius $EO = R$ given in Table I, the tangent at E being, of course, parallel to AC.

Or make $Fm = \frac{1}{2}g$ and perpendicular to FV, that is, make angle $DFm = 90° - F$, and run the curve both ways from m, the tangent at m being parallel to FV.

DOUBLE TURNOUTS FROM A STRAIGHT TRACK.

In case of a double turnout (Fig. 122) three frogs are required; the frogs F and F' being the same as F in Fig. 114, the third frog being at F'', where the lead-rails cross each other.

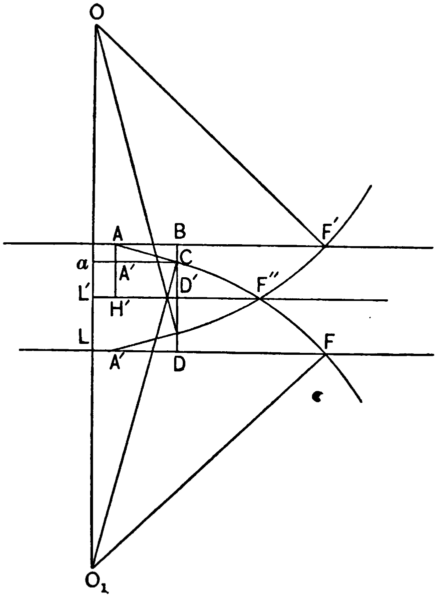


FIG. 122.

Draw the line $H'F''$ parallel to the main track; then it is plain that *one half* the frog-angle F'' is equal to the angle made between the line $H'F''$ and a tangent at F'' to either arc AF or $A'F'$. Therefore the formulas for this case may be written at once from eqs. (2), (3), (4), (5), and (6) by putting $CD' = \frac{1}{2}g - d$ instead of $CD = g - d$, and $\frac{1}{2}F''$ instead of F. Hence

$$R' = \frac{\frac{1}{2}g - d}{\text{vers}\,\frac{1}{2}F'' - \text{vers}\,S}, \quad \text{or} \quad \text{vers}\,\tfrac{1}{2}F'' = \text{vers}\,S + \frac{\frac{1}{2}g - d}{R_1}. \quad . \ (11)$$

$$CF''=\frac{\frac{1}{2}g-d}{\sin\frac{1}{2}(\frac{1}{2}F''+S)}. \qquad . \; . \; . \; . \; . \; . \qquad (12)$$

$$D'F''=(\tfrac{1}{2}g-d)\cot\tfrac{1}{2}(\tfrac{1}{2}F''+S) \qquad H'F''=l+D'F''. \; . \qquad (13)$$

$$L'F''=R'\sin\tfrac{1}{2}F''. \qquad . \; . \; . \; . \; . \; . \; . \qquad (14)$$

Table VI gives the numbers (n''), angles (F''), and distances from head-block of middle frogs corresponding to frogs of different numbers at F and F'.

Example.—Let the frogs at F and F' be number $8\frac{1}{2}$. Then

$$F = F' = 6° \, 44', \quad \text{and} \quad \text{vers } F - \text{vers } S = .006511.$$

Eq. (2) gives

$$R' = \frac{4.2916}{.006511} = 659.14. \quad \therefore \; R = 656.79.$$

Then (11) gives

$$\text{vers } \tfrac{1}{2}F'' = .0003859 + \frac{1.9375}{659.14} = .0035253.$$

$$\therefore \; \tfrac{1}{2}F'' = 4° \, 40' \, 26'', \quad \text{and} \quad F'' = 9° \, 20' \, 52''.$$

From (4),

$$D'F'' = (\tfrac{1}{2}g - d)\cot\tfrac{1}{2}(\tfrac{1}{2}F'' + S) = 1.9375 \times 18.2709 = 35.3998,$$

and

$$H'F'' = 50.3998.$$

Finally, $\quad n'' = \frac{1}{2}\cot\frac{1}{2}F'' = \dfrac{12.2257}{2} = 6.11.$

Hence a number 6 frog at F'', the curves being slightly flattened at the frog F'', will answer the requirement.

To find an approximate relation between F and F''.

Comparing eqs. (2) and (11), omitting d and vers S from both, we have

$$\text{vers } \tfrac{1}{2}F'' = \tfrac{1}{2} \text{ vers } F, \quad \text{or} \quad 4 \text{ vers } \tfrac{1}{2}F'' = 2 \text{ vers } F.$$

But versines of small angles are nearly as the squares of the angles. Hence, approximately,

$$\text{vers } F'' = 4 \text{ vers } \tfrac{1}{2}F''.$$

These equations give

$$\text{vers } F'' = 2 \text{ vers } F, \quad \text{or} \quad (F'')^2 = 2(F)^2, \quad \text{or } F'' = \sqrt{2}F.$$

$$\text{If} \qquad F = F' = 6^\circ\, 44', \quad \text{or} \quad n = n' = 8\tfrac{1}{2},$$

$$\text{then} \qquad n'' = \frac{n}{\sqrt{2}} = 6.01 = 6.0, \text{ nearly}, \quad \text{or} \quad F'' \doteq 9^\circ\, 32', \text{ nearly}.$$

Hence in practice take R from Table VI opposite F, and $F'' = \sqrt{2}F$ as nearly as possible.

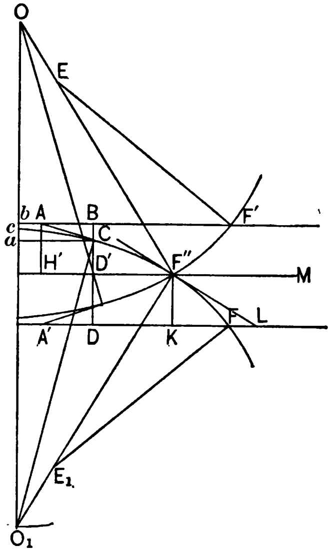


FIG. 123.

In case of a double turnout, where no frog F'' is at hand of the number given by eq. (11) or by $F'' = \sqrt{2}F$, we may locate the turnout as a compound curve, and thus fit the frogs, provided $\frac{1}{2}F''$ is less than F.

Let $OF'' = O_1F'' = R'$, and $EF'' = E_1F'' = R_1$. (Fig. 123.)

R' and the lead $H'F'$ are found as in the preceding problem.

We observe that in Fig. 114 the inclination to the main track of the arc at C is S; while in Fig. 123 the inclination to the main track of the arc at F'' is $\frac{1}{2}F''$.

Moreover, $CD = g - d$; while $F''K = \frac{1}{2}g$. Hence it is evident that we can write at once the relation between F'' and F from the relation between S and F by substituting $\frac{1}{2}F''$ for S, and $\frac{1}{2}g$ for $g - d$ We thus find

$$\text{vers } F = \text{vers } \tfrac{1}{2}F'' + \frac{\frac{1}{2}g}{R'}, \quad \text{or} \quad R' = \frac{\frac{1}{2}g}{\text{vers } F - \text{vers } \frac{1}{2}F''}. \quad (15)$$

Since the versine of the angle of the inclination of the curve cCF at any point to the main track is equal to the distance of that point below c divided by the radius R', it follows that the difference between the versines of such angles at any two points is equal to the difference between their distances below c (or from the rail AB or DF) divided by R'. See eqs. (2) and (15). Thus the relation between the versines at any two points may be written at once.

When $F = \frac{1}{2}F''$, $R_1 = \infty$, showing that the turnout beyond F'' is the straight line $F''L$. This is apparent from the figure, since the angle $KLF'' = LF''M = \frac{1}{2}F''$.

When $F < \frac{1}{2}F''$, R_1 is negative, showing that the turnout must reverse at or beyond F'' in order to cross the main track at the required angle.

By prolonging the arcs $F''C$ as in Fig. 114, the equations for the last two cases may readily be deduced in the same way as were those for Fig. 114.

Example.—$F = 6° \, 44'$, $F'' = 8° \, 48'$.

From (11) we find

$$R' = \frac{\frac{1}{2}g - d}{\text{vers}\,\frac{1}{2}F'' - \text{vers}\,S} = \frac{1.9375}{.0025613} = 756.45,$$

and from (13)

$$H'F'' = l + (\tfrac{1}{2}g - d)\cot\tfrac{1}{2}(\tfrac{1}{2}F'' + S)$$

$$= 15 + 1.9375 \times 19.0550 = 51.9191.$$

(15) gives

$$R_1 = \frac{\frac{1}{2}g}{\text{vers}\,F - \text{vers}\,\frac{1}{2}F''} = \frac{2.35416}{.0039502} = 595.96,$$

and, from (4),

$$KF = \tfrac{1}{2}g\cot\tfrac{1}{2}(F + \tfrac{1}{2}F'') = 2.35416 \times 11.2789 = 26.552.$$

FF'' may be found by (3), but is not needed.

Supposing $F'' < 2F$, then we have approximately as follows:

When $F'' = F\sqrt{2}$, $R_1 = R'$, as above pointed out (see Fig. 122).

When $F'' < F\sqrt{2}$, $R_1 < R'$, as shown in Fig. 123.

When $F'' > F\sqrt{2}$, $R_1 > R'$.

TURNOUTS FROM CURVES.

Given the radius R of the center line of the main track, and the frog angle F, to determine the position of the frog and the radius R' of the center line of the turnout.

I. When the turnout is from the inside of the curve. (Fig. 124.)

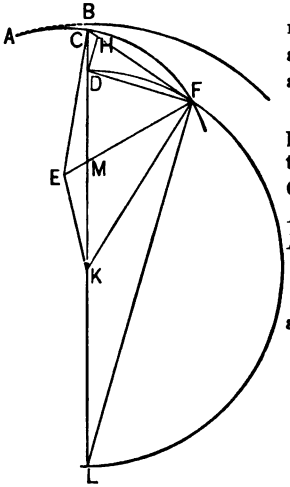


FIG. 124

Let AB and DF be the rails of the main track, AC the switch-rail, the arc CF the outer rail of the turnout, and F the point of the frog.

Since the radii FE and FK are perpendicular to the tangents of the two curves at F, the angle $EFK = F$. Complete the semicircle DFL, join FL, and draw DH parallel to it. Let $FCK = C$. Then

$$ECF = EFC = C + S,$$

and $$CFK = C + F + S.$$

Let $DKF = K$. Then

$$KLF = KFL = CDH = \tfrac{1}{2}K.$$

Now

$$DFK + CFD = CFK = C + F + S,$$

and $$FDK - CFD = DFK - CFD = DCF = C.$$

Subtracting and dividing by 2 gives

$$CFD = \tfrac{1}{2}(F + S).$$

We have

$$\cot \tfrac{1}{2}K = \frac{LF}{DF}, \quad \text{and} \quad \tan \tfrac{1}{2}(F + S) = \frac{DH}{DF}.$$

$$\therefore \frac{\cot \frac{1}{2}K}{\tan \frac{1}{2}(F + S)} = \frac{LF}{DH} = \frac{CL}{CD} = \frac{2R - d}{g - d},$$

or $$\cot \tfrac{1}{2}K = \tan \tfrac{1}{2}(F + S)\frac{2R - d}{g - d}. \quad . \quad . \quad . \quad . \qquad (18)$$

Again, $$CF = \frac{CD \sin CDF}{\sin CFD} = \frac{(g-d)\cos \frac{1}{2}K}{\sin \frac{1}{2}(F + S)}. \qquad (19)$$

Also,

$$CMF = F + K, \quad \text{and} \quad CEF = CMF - ECM = F + K - S,$$

and we have

$$CE = R' + \tfrac{1}{2}g = \frac{\tfrac{1}{2}CF}{\sin\tfrac{1}{2}(F + K - S)} \quad . \quad . \quad (20)$$

II. If the turnout is from the outside of the curve (Fig. 125), complete the semicircle, join FL' and draw DH parallel to it to meet FC prolonged.

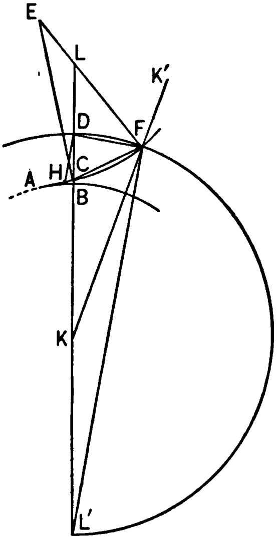


FIG. 125.

Now $EFK' = F$.

Let $DCF = C$.

Then $ECF = EFC = C + S$.

$\therefore\ CFK' = C + F + S$.

$DFK' + CFD = CFK' = C + F + S$,

and

$DFK' - CFD = FDL - CFD = C$.

Subtracting, and dividing by 2, gives

$$CFD = \tfrac{1}{2}(F + S).$$

Let $DKF = K$.
Then

$$KL'F = KFL' = CDH = \tfrac{1}{2}K.$$

We have

$$\cot\tfrac{1}{2}K = \frac{FL'}{DF}, \quad \text{and} \quad \tan\tfrac{1}{2}(F + S) = \frac{DH}{DF}.$$

$$\therefore\ \frac{\cot\tfrac{1}{2}K}{\tan\tfrac{1}{2}(F + S)} = \frac{FL'}{DH} = \frac{CL'}{CD} = \frac{2R + d}{g - d},$$

or

$$\cot\tfrac{1}{2}K = \tan\tfrac{1}{2}(F + S)\frac{2R + d}{g - d}. \quad . \quad . \quad (18')$$

Then

$$CF = \frac{CD\sin CDF}{\sin CFD} = \frac{(g - d)\cos\tfrac{1}{2}K}{\sin\tfrac{1}{2}(F + S)}. \quad . \quad . \quad (19')$$

Again, $CEF = CLF - ECL$, and $CLF = F - K$.

$\therefore\ CEF = F - K - S$.

Finally,

$$R' + \tfrac{1}{2}g = EF = \frac{\frac{1}{2}CF}{\sin\frac{1}{2}CEF} = \frac{\frac{1}{2}CF}{\sin\frac{1}{2}(F - K - S)}. \quad (20')$$

Since in both cases K is small, $\cos\frac{1}{2}K = 1$, very nearly, and the lead $= \dfrac{g - d}{\sin\frac{1}{2}(F + S)}$, very nearly, which is the value of the lead in a turnout from a straight track. Hence Table VI applies to turnouts from curves as well as from straight lines.

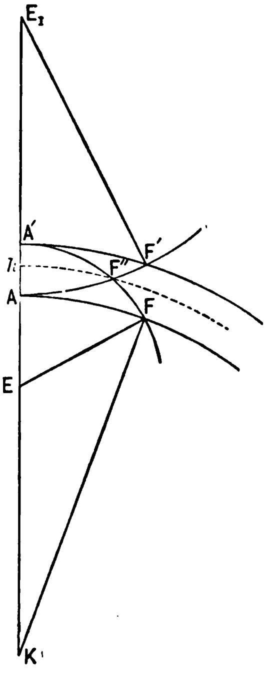


FIG. 126.

In case of a double turnout on opposite sides of a curve, Fig. 126, the frogs F and F', if equal, are opposite, since the chords AF' and $A'F$, as we have just seen, are very nearly equal.

Comparing Figs. 122 and 126, ignoring vers S and d, we see that, in both cases, F' is in the center of the main track, and that the relation between F (or F') and F'' is the same in this case as in the case of a double turnout on opposite sides of a straight track; and consequently, in this case also, we have

$$\text{vers}\,\tfrac{1}{2}F'' = \tfrac{1}{2}\,\text{vers}\,F,$$

or $$\text{vers}\ F'' = 2\,\text{vers}\,F,$$

or $$F'' = \sqrt{2}F, \text{ nearly}.$$

We learn from the preceding discussion and results that frogs in the approximate ratio of $\dfrac{F''}{F} = \sqrt{2}$ are needed in the cases of double turnouts on opposite sides of a straight track or on opposite sides of a curved track. Such frogs should therefore be kept on hand, Thus if F corresponds to a number $8\frac{1}{2}$ frog, then F''' corresponds to a number 6 frog, nearly.

Problem.—To find an approximate value of the degree of a turnout from a curve in terms of the degree of a turnout from a straight line, the turnouts to fit the same frog, F.

Let $d =$ the degree of the center line of the main track, $D =$ the degree of the center line of the turnout from the main track, and $D_1 =$ the degree of the center line of the turnout from a straight track to fit the frog F, as in Fig. 114.

Let $l =$ the length of either arc CF or DF in Figs. 114, 124, and 125. Ignoring the difference in curvature between the center line and either rail of a track, we have the following equations:

In Fig. 114 the difference in the directions of the arc at C and at F is evidently equal to $F - S$. But it is also equal to $\frac{l}{100}D_1$.

Hence
$$\frac{l}{100}D_1 = F - S. \quad . \; . \; . \; . \; . \; . \; . \quad (21)$$

In Fig. 124, since the arcs CF and DF make an angle of S with each other at C and D, and an angle of F with each other at F, it is evident that the change in direction of the arc CF from C to F, *less* the change in direction of the arc DF from D to F, is equal to $F - S$. But the change in $CF = \frac{l}{100}D$, and that of $DF = \frac{l}{100}d$.

Hence
$$\frac{l}{100}D - \frac{l}{100}d = F - S.$$

Equating with the above and dividing by $\frac{l}{100}$, we find

$$D - d = D_1, \quad \text{or} \quad D = D_1 + d. \quad . \; . \; . \; . \quad (22)$$

In Fig 125 the change in direction from C to F *plus* the change from D to F is equal to $F - S$. Hence

$$\frac{l}{100}D + \frac{l}{100}d = F - S.$$

Equating, we find

$$D + d = D_1, \quad \text{or } D = D_1 - d. \quad . \; . \; . \; . \quad (23)$$

Hence, to find the degree of a turnout for a given frog from the *inside* of a curve: To the degree of a turnout for the same frog from a straight track, *add* the degree of the *given* track.

To find the degree of a turnout for a given frog from the *outside* of a curve: From the degree of a turnout for the same frog from a straight track, *subtract* the degree of the *given* track.

Example 1.—Required the degree of a turnout from the inside, as well as from the outside, of a 4° curve for a No. 9 frog.

The degree of a turnout from a straight track for a No. 9 frog is 7° 44′. Hence the turnout from the inside is 11° 44′, and from the outside is 3° 44′. (See Figs. 124 and 125.)

Example 2.—Required the degree of a turnout from the outside of a 7° 44′ curve for a No. 9 frog.

In this case the degree of the turnout is 7° 44′ less 7° 44′, or 0. Hence the turnout is a straight track. In this case the turnout may be regarded as the main track, and the main track as the turnout. (See Fig. 114.)

Example 3.—Required the degree of a turnout from the outside of a 11° curve for a No. 10 frog.

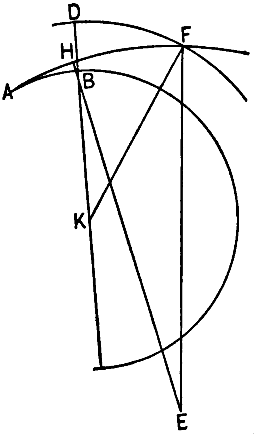


Fig. 127.

In this case the degree of the main track *exceeds* the degree of a turnout from a straight track for the given frog by 11° − (6° 09′) = 4° 51′. Hence, referring to Fig. 127, we see that in this case the center E_1 of the turnout, instead of being on the outside of the main track as in Example 1, is on the inside, that is, on the same side as K.

Example 4.—Let F (in Fig. 124) = 5° 43′. Main track an 8° curve, or $r = 716.197$. $S = 1°\ 35'$. Required the lead CF, and the radius of the center line of the turnout. We have

$$CF = \frac{(g-d)\cos\frac{1}{2}K}{\sin\frac{1}{2}(F+S)} = \frac{4.2916}{.06366} = 67.54.$$

The degree of turnout curve from a straight track, corresponding to the frog F, is, by Table VI, 6° 09′ +.

Adding the degree of the main track (8°), we have

$$\text{degree of turnout} = 14° \, 09' +,$$

and the radius is $R = 404.92 -$.

Or (18) gives

$$\tan KDF = \frac{(2R - d) \tan \frac{1}{2}(F + S)}{g - d} = \frac{1431.98 \times .06379}{4.2916} = 21.2848.$$

$$\therefore \quad KDF = 87° \, 18' \, 36''. \qquad \therefore \quad \tfrac{1}{2}K = 2° \, 41' \, 24''.$$

Then (19) gives

$$CF = \frac{(g - d) \cos \frac{1}{2}K}{\sin \frac{1}{2}(F + S)} = 67.34.$$

Then, by (20), we find

$$R' + \tfrac{1}{2}g = \frac{\frac{1}{2}CF}{\sin \frac{1}{2}(F + K - S)} = \frac{33.67}{.08293} = 406.01,$$

and $R' = 403.66$, and $D = 14° \, 11' +$.

The above approximate results differ but little from the true results and show that the approximate formulas are practically all-sufficient.

OTHER TURNOUTS FROM A STRAIGHT TRACK.

From the same point in a straight track it is required to locate two turnouts on the same side. (Fig. 128.)

We will assume that $F = F'$, and that these frogs are opposite to each other, and that the curves are tangent to the main track at A.

Let $KA = r + \frac{1}{2}g$, and $EA = r' + \frac{1}{2}g$.

The angle $AKF = F$, and the angle $KF'E = F' = F$.

Hence the triangle KEF' is isosceles. Therefore

$$EK = EF' = AE,$$

or $AE = \frac{1}{2}AK$.

That is,

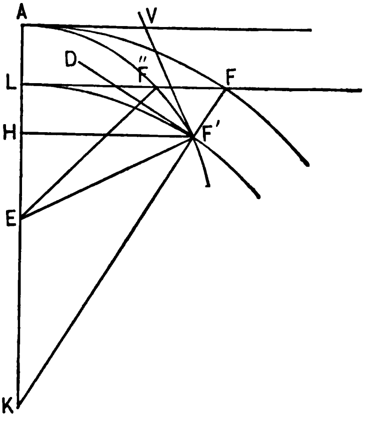


FIG. 128.

$$r' + \tfrac{1}{2}g = \tfrac{1}{2}(r + \tfrac{1}{2}g), \quad \text{or} \quad r' = \tfrac{1}{2}(r - \tfrac{1}{2}g). \quad . \quad . \quad (24$$

This relation is easily proved as follows:

Let d = the degree of the curve AF, and d' that of AF'. Now we have seen that $d' = d +$ the degree of a turnout from a straight track for the frog F, i.e., $d' = d + d$, or

$$\frac{1}{r' + \frac{1}{2}g} = 2\frac{1}{r + \frac{1}{2}g}, \quad \text{or} \quad r' = \tfrac{1}{2}(r - \tfrac{1}{2}g).$$

Or again: Draw the tangents VF' and DF'; also HF' parallel to LF. Evidently,

$$VF'D = F' = F, \quad \text{and} \quad DF'H = F.$$

$$\therefore \quad VF'H = 2F.$$

Hence

$$\text{vers } VF'H = \text{vers } AEF' = \text{vers } 2F = \frac{AH}{AE} = \frac{AH}{r' - \frac{1}{2}g}.$$

Also,

$$\text{vers } F = \frac{AL}{AK} = \frac{\frac{1}{2}AH}{r + \frac{1}{2}g}; \quad \text{vers } 2F = 4 \text{ vers } F = \frac{AH}{\frac{1}{2}(r + \frac{1}{2}g)}.$$

$$\therefore \quad r' + \tfrac{1}{2}g = \tfrac{1}{2}(r + \tfrac{1}{2}g), \text{ as before.}$$

In this we assume $LH = AL = F'F$; but $LH = F'F \cos F$. We also assume vers $2F = 4$ vers F; but vers $2F < 4$ vers F. The two *small* errors exactly balance, however, and result in a true formula.

We have $$\text{vers } F' = \text{vers } F = \frac{g}{r + \frac{1}{2}g}, \quad . \; . \; . \; . \quad (25)$$

and $$\text{vers } F'' = \frac{g}{r' + \frac{1}{2}g}.$$

But $$r' + \tfrac{1}{2}g = \tfrac{1}{2}(r + \tfrac{1}{2}g).$$

$$\therefore \quad \text{vers } F'' = 2 \text{ vers } F, \quad \text{or} \quad F'' = \sqrt{2}F, \text{ nearly.}$$

This is the same relation between F and F'' as was above shown to exist between the frogs in the case of turnouts to the opposite sides of a straight or of a curved track. (Figs. 122 and

126.) Hence a set of frogs adapted to a double turnout on opposite sides of a straight or a curved track is also adapted to such a turnout on the same side of a straight track. In case of the double turnout on the same side of a straight track, the longer radius is equal to twice the shorter radius $+ \frac{1}{2}g$.

In case no frog is at hand equal to F'' given by the relation vers $F'' = 2$ vers F, we may select one, which call F'', as near the same angle as possible, and find, as already shown (see Fig. 129), $EF'' = r' + \frac{1}{2}g = \frac{g}{\text{vers } F''}$, and $BF'' = (r' + \frac{1}{2}g) \sin F''$. Then compound the curve at F'', and find the radius OF' to suit the frog F', whether it is equal or unequal to F.

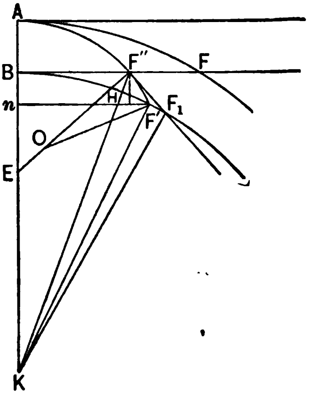


FIG. 129.

Let $AK = r + \frac{1}{2}g$ as before.

Then $KF' = r - \frac{1}{2}g$.

Assume

$$OF' = r'' + \tfrac{1}{2}g.$$

Then $$EO = r' - r'', \quad \text{and} \quad EK = r - r'.$$

$$BEF'' = F'', \quad \text{and} \quad OF'K = F'.$$

Let $F'KF'' = K$. We have

$$\tan BKF'' = \frac{BF''}{BK} = \frac{BF''}{r - \frac{1}{2}g}. \quad . \quad . \quad . \quad . \quad (27)$$

Then $OF''K = BEF'' - BKF'' = F'' - BKF'' = A$, say.

Again,

$$F''F'K = F''F'O + OF'K = F''F'O + F',$$

and $$F'F''K = F'F''O - OF''K = F''F'O - A.$$

Subtracting gives

$$F''F'K - F'F''K = A + F'.$$

Also,

$$KF'' = \frac{BF''}{\sin BKF''}, \text{ and } HF'' = KF'' - (r - \tfrac{1}{2}g) = e, \text{ say.} \quad (28)$$

Hence

$$KF'' + KF = 2(r - \tfrac{1}{2}g) + e, \quad \text{and} \quad KF'' - KF = e.$$

Applying the "tangent proportion" to the triangle $KF'F''$, we have

$$e : 2(r - \tfrac{1}{2}g) + e :: \tan \tfrac{1}{2}(A + F') : \cot \tfrac{1}{2}K,$$

or
$$\cot \tfrac{1}{2}K = \left(\frac{2r - g + e}{e}\right) \tan \tfrac{1}{2}(A + F'). \quad . \quad . \quad (29)$$

Now

$$BKF' = BKF'' + K, \quad \text{and} \quad BF' = 2(r - \tfrac{1}{2}g) \sin \tfrac{1}{2}BKF'. \quad (30)$$

This gives the position of F'.

We have

$$\tfrac{1}{2}(KF'F'' + KF''F') = 90° - \tfrac{1}{2}K,$$

and
$$\tfrac{1}{2}(KF'F'' - KF''F') = \tfrac{1}{2}(A + F'), \text{ just found.}$$

Subtracting gives

$$KF''F' = 90° - \tfrac{1}{2}(A + F' + K).$$

Now
$$F''F' = \frac{F'K \sin K}{\sin KF''F'}. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Adding $OF''K = A$ to the above, we have

$$OF''F' = 90° + \tfrac{1}{2}A - \tfrac{1}{2}(F' + K).$$

Taking $OF''F' + OF'F'' = 2OF''F'$ from 180°, we have

$$F'OF'' = F' + K - A.$$

Finally,
$$OF' = r'' + \tfrac{1}{2}g = \frac{\tfrac{1}{2}F''F'}{\sin \tfrac{1}{2}F'OF''}. \quad . \quad . \quad . \quad . \quad (32)$$

Suppose the turnout straight beyond F'', the frog being at F_1. In the triangle F_1KF''

$$\text{angle } F_1F''K = 90^\circ - OF''K = 90^\circ - A.$$

$$\therefore \quad \sin F_1F''K = \cos A.$$

Also, $KF_1 = r - \frac{1}{2}g$, and $KF'' = r - \frac{1}{2}g + e$.

Hence $$\sin KF_1F'' = \frac{r - \frac{1}{2}g + e}{r - \frac{1}{2}g} \cos A.$$

$$F_1KF'' = 180^\circ - (KF_1F'' + KF''F_1).$$

Then $$F''F_1 = KF_1 \frac{\sin F_1KF''}{\sin F_1F''K} = \frac{(r - \frac{1}{2}g) \sin F_1KF''}{\cos A}. \quad . \quad (33)$$

This gives the position of the frog F_1.

Example.—$F = 6^\circ\ 43'\ 59''$, $F' = 6^\circ\ 01'\ 32''$, $F'' = 8^\circ\ 47'\ 51''$.

By Table VI, $BF = 80.036$, $r = 680.306$;

$BF'' = 61.204$, $r' = 397.826$.

By (27),

$$\tan BKF'' = \frac{61.204}{677.952} = .0902777. \qquad \therefore \quad BKF'' = 5^\circ\ 07'\ 31''.$$

$$A = F'' - BKF'' = 3^\circ\ 38'\ 20''; \qquad A + F' = 9^\circ\ 39'\ 52'';$$

$$\tfrac{1}{2}(A + F') = 4^\circ\ 49'\ 56''.$$

By (28),

$$KF'' = \frac{61.204}{.089913} = 680.702. \qquad \therefore \quad e = 680.702 - 677.952 = 2.75.$$

Eq. (29) gives

$$\cot \tfrac{1}{2}K = \left(\frac{2r - g + e}{e}\right) \tan \tfrac{1}{2}(A + F') = 494.056 \times .0845388 = 41.76691.$$

$$\therefore \quad \tfrac{1}{2}K = 1^\circ\ 22'\ 42'', \quad \text{and } K = 2^\circ\ 45'\ 24''.$$

$$KF''F' = 90^\circ - \tfrac{1}{2}(A + F' + K) = 83^\circ\ 47'\ 22''.$$

Then, by (31),

$$F''F' = \frac{(r - \frac{1}{2}g)\sin K}{\sin KF''F'} = \frac{677.952 \times .048943}{.994131} = 33.798.$$

$$\tfrac{1}{2}(F' + K - A) = 2° \, 34' \, 18''.$$

Then (32) gives

$$r'' + \tfrac{1}{2}g = \frac{\frac{1}{2}F''F'}{\sin \frac{1}{2}(F' + K - A)} = \frac{16.999}{.044869} = 365.486,$$

and $$r'' = 363.132.$$

CHAPTER IX.

THE TRUE TRANSITION CURVE.

THE object of this chapter is to make known the *true* transition curve, to show its need, to furnish *simple* formulas for its use, and to show its ready application in practice.

FUNDAMENTAL PRINCIPLES.

When a car passes from a straight line upon a curve, or *vice versa*, it receives a shock more or less severe, in proportion to the sharpness of curvature and the rate of speed. This, it is well known, is due to a tendency of any body in motion to persist in its direction of motion at every point of its path.

This shock is damaging to rolling stock and track, causes discomfort to travelers, and is, moreover, a source of danger.

It is evident that the elevation of the outer rail cannot obviate *this* difficulty to any appreciable extent; for it is plain that such elevation, which must vary directly with the degree of curvature (see Elevation of Rail, Chap. V), can correspond to a *gradual* change of curvature only, and not to a sudden change, like passing from a straight line upon a curve or the reverse.

Whatever, therefore, the elevation might be, a car passing from a tangent upon a sharp curve, or conversely, would receive a violent shock. The only way possible to do away with such shocks and the resulting evils is to interpose between the tangent and the main curve a "transition curve."

A transition curve, as its name indicates, is a curve placed between a tangent and a main curve.

A theoretically perfect transition curve must have the following properties:

1. It must end on the curve with a radius equal to that of the curve, and on the tangent in a straight line.

2. The degrees of curvature of the transition curve at different points must vary directly as their distances from its junction with the tangent.

To these must be added the following practical requisites:

3. The curve should be one easy to understand and especially easy to lay out.

4. It should be flexible, so as to accommodate itself to the configuration of the ground and to other conditions.

Since the curvature of the transition curve is less than that of the main curve with which it connects except at and near the point of tangency, it must lie outside of the latter prolonged.

Hence, when a curve is located ending in a tangent, it is necessary either to move the curve inward or the tangent outward, in order to interpose a transition curve between them. We will suppose the curve to be moved.

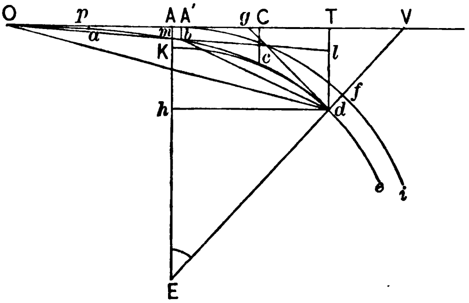


FIG. 130.

Thus, referring to Fig. 130, *AO* represents a tangent and *A* the tangent point of the curve *Afi*, which is replaced by the slightly sharper concentric curve *kde*. This latter curve is connected with the original tangent at *O* by the transition curve *Omd*, *m* being the intersection of that curve with *AK*, which is the distance between the concentric curves, and is called the offset.

Since the curves *Kd* and *Omd* subtend the same central angle, while the latter is flatter than the former it must be longer also, and therefore *O* must be back of *A*.

Again, since the radius of curvature of the transition curve at *d* is equal to *dE*, which is less than *fE*, we see that the curvature of that curve at and near to *d* is a little sharper than that of the original curve *Afi*.

Now the degrees of curvature of different arcs are as the angles ed in passing over equal lengths of those arcs. Hence, in order that the degrees of curvature at different points of *Omd* may vary as their distances from *O*, the angles turned in passing over arcs of the same length, supposed to be very short, must vary as the distances of their centers from *O*; and the total angles turned between *O* and different points must therefore vary as the squares of the distances of those points from *O*.

Hence if *O* is the total angle *dgT* turned in passing over any length of this curve, as *Omd* = *S*, we must have

$$S^2 = cO, \quad c \text{ being a constant. . . . . . } \quad (1)$$

A transition curve should be long enough to allow the outer rail to gain the elevation at *d*, proper for the main curve without inclining too abruptly.

Froude, Rankine, and others allow an incline of one foot in 300; and, since the maximum elevation of the outer rail should not exceed 8 inches or $\frac{2}{3}$ of a foot, transition curves need not exceed $\frac{2}{3} \times 300 = 200$ feet in length. Curves of any length, however, may be readily used by the formulas to be given.*

ELEMENTARY RELATIONS.

By elementary relations is meant those pertaining to lines and angles of the figure.

1. To find the length of the offset curve in terms of the length of the transition curve.

Let $dEK = dgT = O$, and $S =$ the length of the transition curve *Omd*. Then

$$Kd = \frac{S}{2}, \text{ exactly. . . . . . . } \quad (2)$$

The figure shows that *md* is a trifle greater than $Kd = \frac{S}{2}$, and therefore *mO* is a trifle less than $Kd = \frac{S}{2}$.

Also, *AO* is a trifle less than *mO*, hence a trifle less than $\frac{S}{2}$.

Let $dE = R'$, then

$$AT = dh = R' \sin O. \text{ . . . . . . } \quad (3)$$

---

* The author hopes to give the proof of the following formulas in a treatise on the True Transition Curve, to appear later.

2. **To find any tangent distance.**

We find
$$OT = S\left(1 - \frac{O^2}{10} + \frac{O^4}{216} -\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or
$$OT = S \cos \frac{201}{450}O, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or
$$OT = S \cos \frac{4}{9}O, \text{ very nearly.} \quad . \quad . \quad . \quad . \quad (6)$$

Example.—Let $O = \frac{1}{3}$ of the unit angle $= 19^\circ.09859$. Then (4) gives

$$OT = S\left(1 - \frac{1}{90} + \frac{1}{81 \times 216} -\right) = S(1 - .011 + .00005716 -) = .988946S.$$

Eq. (5) gives $OT = .988936S.$
Eq. (6) gives $OT = .989046S.$

Hence (5) gives result too small by only $.00001S$, and (6) a result too large by only $.0001S$.

O in this example is very large and the errors are therefore comparatively very large. Hence (6) is sufficiently exact for all cases.

Let $R' = KE$. Now $Kd = KE \times KEd$. That is,

$$\frac{S}{2} = R'O; \text{ or } R' = \frac{S}{2O}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Again we find

$$AO = \frac{S}{2}\left(1 - \frac{O^2}{30} + \frac{O^4}{1080} -\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or
$$AO = \frac{S}{2}\cos\frac{31}{120}O, \text{ almost exactly,} \quad . \quad . \quad . \quad (9)$$

or
$$AO = \frac{S}{2}\cos\frac{1}{4}O, \text{ very nearly.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For $O = 19^\circ.09859$,

(8) gives $AO = \frac{S}{2}(1 - .00370 + .00001143) = .49815S,$

(9) gives $AO = .49815S,$
and (10) gives $AO = .49826S.$

We observe that (9), even for this very large value of O, gives a result true to five places; and that (10) is also practically exact.

Letting $KE = R'$, we also find

$$AO = \tfrac{9}{4}R' \sin \tfrac{4}{9}O. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

To find the tangent distance corresponding to any part of the curve it is only necessary to substitute in (4), (5), or (6) the corresponding values of the arc and of the angle. Thus suppose $Oab = \tfrac{1}{2}S$. Then the spiral angle for b is $\frac{O}{4}$, and we have

$$OA' = \frac{S}{2}\left(1 - \frac{O^2}{160} + \frac{O^4}{256 \times 216} -\right) = \frac{S}{2} \cos \frac{67}{600}, \text{almost exactly.} \qquad (12)$$

$$= \frac{S}{2} \cos \frac{1}{9}O, \text{ very nearly.} \qquad (13)$$

Now, from (10) and (13),

$$AA' = OA' - OA = \frac{S}{2}(\cos \tfrac{1}{9}O - \cos \tfrac{1}{4}O). \quad . \qquad (14)$$

For $O = 19^\circ.09859$,

$$AA' = .00278\frac{S}{2} = .00139S.$$

This shows the distance of the middle of the curve b to the right of AmK.

For all ordinary values of O this distance can be neglected and the middle of the curve be taken as at m.

From (9) and (10) we have

$$\frac{S}{2} - AO = \frac{S}{2} \text{ vers } \tfrac{21}{120}O = d, \text{ suppose}, \quad . \quad . \quad . \qquad (15)$$

and

$$\frac{S}{2} - AO = \frac{S}{2} \text{ vers } \tfrac{1}{4}O \ = d, \text{ suppose.} \quad . \quad . \quad . \qquad (16)$$

These values are very easily computed, since the versines of the angles contain few significant figures.

Then, d being very small, $AO = \frac{S}{2} - d$ may be easily computed mentally.

3. To find any offset, as dT. We find

$$dT = S\left(\frac{O}{3} - \frac{O^3}{42} + \frac{O^5}{1320} - \right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$\left.\begin{array}{ll} \text{or} & dT = \frac{S}{2} \sin \tfrac{2}{3}O + = \frac{S}{2} \sin \tfrac{2}{3}O, \text{ very nearly.} \\ \text{Also,} & dT = dV \cos O. \end{array}\right\} \quad (18)$$

For $O = 19^\circ.09859$, (17) gives

$$dT = .11023S,$$

and (18) gives

$$dT = .11021S,$$

which is sufficiently accurate even for this very large value of O.

The offset for any point of the curve is readily found by substituting in (17) or (18) the arc and the angle corresponding to the point.

Thus let c = arc Om, and E = the corresponding spiral angle.

$$\text{Then} \qquad Am = c\left(\frac{E}{3} - \frac{E^3}{42} + \frac{E^5}{1320} - \right), \quad . \quad . \quad . \quad . \quad (19)$$

$$\text{or} \qquad Am = t = \frac{c}{2} \sin \frac{2E}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Eq. (1) shows that E varies as c^2. Also, sin $\tfrac{2}{3}E$ increases less rapidly than E, and therefore less rapidly than c^2.

Hence t increases less rapidly than c^3.

Again, let b be the middle point of the curve. Then $Ob = \frac{S}{2}$, and the angle subtended by the arc $Oab = \frac{O}{4}$.

Substituting these values for S and O in (17), or for c and E in (19), we have

$$A'b = \frac{S}{2}\left(\frac{O}{4.3} - \frac{O^3}{64.42} + \right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

or

$$A'b = \frac{S}{4}\sin\frac{O}{6}, \text{ almost exactly.} \quad . \quad . \quad . \qquad (22)$$

The error of (22) is less than .0000005 for $O = 19°.09857$.

Eq. (7) gives

$$R' = \frac{S}{2O}.$$

Also,

$$\text{vers } O = \frac{O^2}{2}\left(- \frac{O^4}{24} + \frac{O^6}{720} - \right).$$

$$\therefore\ hK = R' \text{ vers } O = S\left(\frac{O}{4} - \frac{O^3}{48} + \frac{O^5}{1440} - \right), \quad . \qquad (23)$$

or

$$hK = \frac{S}{3}\sin \tfrac{3}{4}O, \text{ very nearly,} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

or

$$hK = \tfrac{5}{14}S\sin \tfrac{7}{10}O, \text{ almost exactly.} \quad . \quad . \quad . \quad . \qquad (25)$$

Now from (17) and (23) we get

$$t_1 = AK = dT - hK = S\left(\frac{O}{12} - \frac{O^3}{336} + \frac{O^5}{9360} - \right). \quad . \qquad (26)$$

Hence

$$AK = t_1 = \frac{S}{6}\sin\frac{O}{2} + = \frac{S}{6}\sin\frac{O}{2}, \text{ very nearly,} \quad . \qquad (27)$$

or

$$AK = t_1 = \tfrac{2}{11}S\sin \tfrac{11}{24}O, \text{ with great accuracy.} \quad . \quad . \quad . \qquad (28)$$

Comparing (18), (22), and (27), we see that

$$A'b > \tfrac{1}{2}AK, \quad A'b > \tfrac{1}{8}dT, \quad \text{and} \quad AK > \tfrac{1}{4}dT.$$

However, when O is not very large, we have

$$\left.\begin{array}{l} Ab = \frac{1}{3}AK = \frac{1}{5}dT, \text{ very nearly;} \quad . \quad . \quad . \quad (29) \\ Am = \frac{1}{3}AK = \frac{1}{3}dT, \text{ very nearly.} \quad . \quad . \quad . \quad (30) \end{array}\right.$$

From (10) and (28) we have

$$\frac{AK}{AO} = \frac{\frac{S}{6}\sin\frac{1}{2}O}{\frac{S}{2}\cos\frac{1}{2}O} = \frac{\frac{S}{6}\times 2\sin\frac{1}{4}O\cos\frac{1}{4}O}{\frac{S}{2}\cos\frac{1}{2}O} + = \frac{2}{3}\sin\frac{O}{4}, \text{ very nearly.} \quad (31)$$

The above values are all very simply expressed in terms of S and O, the offset curve being known.

We now require the following:

4. Having an offset t corresponding to an *offset curve* Kd, of length $\frac{S}{2}$ and degree D, to find the change in the offset, or the new offset, t', for the same length of offset curve, when the *original* Afi is a D degree curve.

We have, from (28), $AK = t = \frac{S}{6}\sin\frac{O}{2}$, when Kd is a D degree curve of length $\frac{S}{2}$, the central angle being O.

When, however, Afi is a D degree curve, Kd is a curve of degree $D\frac{AE}{KE} = D\frac{R}{R - t'} = D'$, and, its length remaining the same, the central angle $= O\frac{R}{R - t'} = O'$, and the offset $t' > t$.

We find

$$t' = t + \frac{t^2}{R - 2t'} = t + \frac{t^2}{R - 2t}, \text{ very nearly.} \quad . \quad (32)$$

Or we may find t' as follows:

Suppose, for a moment, Kd, instead of Af, to be a D degree curve of radius R, and compute t or take it from Table XV. Then we have $KE = R - t = R'$, and of course D', with which we may compute t' very accurately, or we can take it from Table XV.

5. To find the offset $AK = t$ in terms of the central angle O and the radius $KE = R'$ or of $AE = R$.

Substituting $2OR'$ for S in (27) and dividing by R' gives

$$\frac{AK}{R'} = \left(\frac{O^2}{6} - \frac{O^4}{168} + \frac{O^6}{4680} - \right). \quad . \quad . \quad . \quad . \quad (33)$$

Represent the series by e; then

$$\frac{t}{R'} = e, \quad \text{or} \quad t = R'e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

Substituting $R - t$ for R' in (34), we find

$$t = R\frac{e}{1+e}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

But $\quad e = \frac{3}{4} \text{ vers } \frac{2}{3}O + = \frac{3}{4} \text{ vers } \frac{2}{3}O$, very nearly. . . (36)

Hence, from (34) and (36),

$$t = R'\tfrac{3}{4} \text{ vers } \tfrac{2}{3}O, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

and, from (35) and (36),

$$t = \frac{R \text{ vers } \frac{2}{3}O}{\frac{4}{3} + \text{vers } \frac{2}{3}O}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

6. To find the angle between the tangent AO and any radius vector or chord drawn from O.

We find

$$\frac{dT}{OT}\frac{d}{t} = \tan AOd = \frac{O}{3} + \frac{O^3}{105} + \frac{O^5}{6000}, \text{ very nearly}, \quad . \qquad (39)$$

Omd representing any arc.

$$\therefore \quad AOd < \frac{O}{3}, \quad \text{but } AOd = \frac{O}{3}, \text{ very nearly.} \quad . \qquad (40)$$

The angle $\frac{O}{3}$ between AO and the radius vector Od may be called the polar angle.

Again,

$$\frac{t}{d} = \tan AOd = \tfrac{8}{7} \tan \tfrac{7}{24}O \text{ with great accuracy.} \quad . \qquad (41)$$

Example.—Let $O = 6° \, 48'$; then $\frac{7}{24}O = 1° \, 59'$.

The formula gives

$$\tan AOd = \tfrac{8}{7} \tan 1° \; 59' = .0395766,$$

$$\therefore \quad AOd = 2° \; 15' \; 59'',$$

which is but a single second less than $\frac{O}{3}$.

The series (39) would give precisely the same result.

Furthermore, $\frac{O}{3} - AOd$ varies as O^3, nearly.

Hence for $O = 13° \; 36' \quad AOd = \frac{O}{3} - 08''$, nearly.

Referring to (39), we see that $\frac{t}{d}$ increases more rapidly than O, and therefore more rapidly than S^2, and still more rapidly than d^2. Hence t increases more rapidly than d^3. For all values of E or O not very large, it is plain, however, that t increases nearly with d^3.

In the cubic parabola whose equation is $x^3 = a^2y$, AO in the figure being the axis of x, the offsets do increase as the cubes of their distances from O, measured along AO, increase.

For all values of O, not large, such curve fulfills the theoretical requirements of a transition curve with reasonable exactness. As O increases, however, the curve departs more and more from the true transition curve and becomes more and more unsuitable for a transition curve.

Moreover, this curve has a minimum radius of curvature which cannot be passed in using it as a transition curve, and which would be troublesome to many. Furthermore, the deflection angles are fractional, are subject to no simple law, and therefore require special computation, which is not the case, as we shall see, with the true transition curve.

7. To find the point of the curve *Omd* where the curve is parallel to the chord *Od*.

Let p be the point. Since at O the curve makes the angle $\tfrac{1}{3}O$ with Od, it changes direction $\tfrac{1}{3}O$ between O and p. But it changes direction an amount O between O and d. Now since these changes are proportional to the squares of the distances Op and Od, we have

$$\frac{\overline{Op}^2}{\overline{Od}^2} = \frac{\frac{1}{3}O}{O} = \frac{1}{3}.$$

$$\therefore\ Op = \sqrt{\tfrac{1}{3}}Od = .57735Od \quad \text{or,} \quad Op = \tfrac{4}{7}Od, \text{ nearly.}$$

8. To find the tangents dg and Og.

We have $\qquad dOg = \frac{1}{3}O,$

and hence $\qquad \therefore\ Odg = \frac{2}{3}O,$ very nearly.

Also, from (18), $\quad dT = \frac{S}{2} \sin \frac{2}{3}O.$

$$\therefore\ dg = \frac{dT}{\sin O} = \frac{S}{2}\,\frac{\sin \frac{2}{3}O}{\sin O}. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

9. To find the length of any radius vector or chord Od and the angles between these chords.

We have

$$\text{angle } AOd = \frac{O}{3},$$

and

$$dT = \frac{S}{2} \sin \tfrac{2}{3}O.$$

$$\therefore\ Od = dT \div \sin\frac{O}{3} = \frac{S}{2}\,\frac{\sin \frac{2}{3}O}{\sin \frac{1}{3}O} = S \cos \frac{O}{3}. \quad . \quad (43)$$

Then $\qquad S - Od = S \text{ vers } \frac{O}{3}. \quad . \quad . \quad . \quad . \quad . \quad (44)$

Hence any radius vector is equal to the arc it subtends multiplied by the cosine of its polar angle.

Let c = arc Oa = arc ab, etc.; these arcs being sufficiently short not to differ sensibly from their chords.

Let θ = spiral angle of Oa, 4θ = spiral angle of Ob, etc. Then

$$\text{chord } Oa = c \cos \frac{\theta}{3}, \quad \text{or} \quad \text{arc } Oa - \text{chord } Oa = c \text{ vers } \frac{\theta}{3},$$

$$\text{chord } Ob = 2c \cos \frac{4\theta}{3}, \text{ or arc } Ob - \text{chord } Ob = 2c \text{ vers } \frac{4\theta}{3}, \text{ etc.}$$

Example.—Let $\theta = 30'$ and $c = 100$.

Since Oa subtends an angle θ, it would subtend an angle 2θ if the curvature was everywhere the same as at a; and since $Oa = 100$, the curvature at a is therefore 1°.

Since Ob subtends an angle of 4θ, and would subtend an angle of 8θ if the curvature were everywhere the same as at b, and since $Ob = 200$, the curvature at b is

$$\frac{8\theta}{2} = 4\theta = 2^\circ.$$

Similarly we find curvature at $c = 3^\circ$, etc.

Or, since at a the curvature $= 1^\circ$, at b, c, etc., it is 2°, 3°, etc.

Then the chord $Oa = 100 \cos 10' = 99.99958$;
the chord $Ob = 200 \cos 40' = 199.986$;
the chord $Oc = 300 \cos 90' = 299.897$;
the chord $Od = 400 \cos 160' = 399.567$, etc.

We observe that, as θ, 4θ, 9θ, etc., increase as 1^2, 2^2, 3^2, etc., the versines of these angles increase as 1^4, 2^4, 3^4, nearly; and $c \text{ vers } \frac{\theta}{3}$, $2c \text{ vers } \frac{4\theta}{3}$, etc., increase as 1^5, 2^5, 3^5, etc., nearly.

We can also find the long chords Ob, Oc, etc., from Table IV opposite the degrees of curvature of the arcs Ob, Oc, etc., at the points where they are parallel to their chords. Thus the degree of curvature for arc Ob is

$$120' \times .57735 = 69'.28;$$

then chord $Ob = 199.986$.

For arc Oc it is $180' \times .57735 = 103'.92$;

then chord $Oc = 299.897$.

For arc Od it is $240' \times .57735 = 138'.56$;

then chord $Od = 399.568$.

We have

$$AOa = \frac{\theta}{3};\quad AOb = \frac{4\theta}{3};\quad AOc = \frac{9\theta}{3};\quad AOd = \frac{16\theta}{3}, \text{ etc.}$$

Then, by subtraction,

$$aOb = \frac{4\theta - \theta}{3} = \frac{3\theta}{3},\ bOc = \frac{9\theta - 4\theta}{3} = \frac{5\theta}{3},\ cOd = \frac{16\theta - 9\theta}{3} = \frac{7\theta}{3}, \text{ etc.}$$

Hence these angles between the chords are in the ratio of the odd numbers 1, 3, 5, 7, etc.

10. To find any deflection angle, as *dbl*, at any point *b*; also any chord *bd*.

bl being the prolongation of *Ob*, we have

$$\tan dbl = \frac{\sin bOd}{\cos bOd - \frac{bO}{dO}}.$$

Then $Odb = dbl - bOd$ and *bd* may be found by the sine proportion.

Or as follows: Let c and c_1 represent the chords *Ob* and *bd*, and S and S_1 the corresponding arcs. Let $Oa = 1$, and $\theta =$ the spiral angle for *Oa*. Then

$$AOd = (S + S_1)^2\frac{\theta}{3}, \qquad AOb = S^2\frac{\theta}{3},$$

and hence $$bOd = S_1(2S + S_1)\frac{\theta}{3}. \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Now $$\sin bdO = \frac{bO}{bd} \sin bOd = \frac{c}{c_1} \sin S_1(2S + S_1)\frac{\theta}{3}.$$

Now $$\frac{c}{c_1} = \frac{S}{S_1}, \text{ nearly.}$$

$$\therefore\ bdO = \frac{S}{S_1} \cdot S_1(2S + S_1)\frac{\theta}{3} = S(2S + S_1)\frac{\theta}{3}, \text{ nearly.} \quad (46)$$

This amounts to assuming that the angles *bOd* and *bdO* are in the ratio of the arcs *bd* and *bO*, which is very nearly true.

Now $$dbl = bOd + bdO = (S + S_1)(2S + S_1)\frac{\theta}{3}. \quad . \quad . \quad (47)$$

We may find any chord, as *bd*, with sufficient accuracy from Table IV, opposite the degree of curvature of the middle of the arc *bd*.

11. To find the exsec dV, also TV, etc.

$$dV = dT \sec TdV = \frac{S}{2} \sin \tfrac{1}{3}O \sec O,$$

or

$$dV = EV - Ed = R \sec O - R';$$

$$TV = dT \tan O = \frac{S}{2} \sin \tfrac{1}{3}O \tan O;$$

$$OV = OT + TV.$$

12. To find the radius of curvature at any point of the curve *Omd*.

We have

$$R' = \frac{Kd}{O} = \frac{S}{2O}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Since R' is a radius of the transition curve expressed in terms of the arc *Omd* and the angle dgT, the relation is perfectly general.

Eliminating O between (1) and (7) and omitting the accent, we have

$$R = \frac{c}{2S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This shows that the *degree* of curvature varies directly with the length of arc measured from O, which agrees with the definition of the curve.

13. To lay out the curve by ordinates, or offsets from the tangent AO.

We have, by (18),

$$ap = \tfrac{1}{3}Oa \sin 2AOa;$$

$$A'b = \tfrac{1}{3}Ob \sin 2AOb;$$

$$cC = \tfrac{1}{3}Oc \sin 2AOc, \text{ etc.}$$

Let the spiral angle of $Oa = 30'$. Then $AOa = 10'$, $AOb = 40'$, $AOc = 90'$, etc. Hence

$$ap = 50 \sin 20' = .291;$$

$$bA' = 100 \sin 1^\circ 20' = 2.33;$$

$$cC = 150 \sin 3^\circ 00' = 7.85, \text{ etc.}$$

Now set a 100 feet from O and .29 of a foot from AO;

b 100 feet from a and 2.33 feet from AO;

c 100 feet from b and 7.85 feet from AO, etc.

We also have, from (6),

$$Op = Oa \cos \tfrac{4}{3} AOa;$$

$$OA' = Ob \cos \tfrac{4}{3} AOb, \text{ etc.}$$

But these quantities are not needed.

Special Problems and Examples.

The reader will observe that since $Kd = \frac{Omd}{2}$, when either Kd or Omd is known the other is known also.

Problem 1.—Given the length and degree of Kd, Fig. 130, to find the offset AK and tangent distance AO, and to lay out the curve.

Example 1.—Let Kd be a 9° 36′ curve 100 feet long.

$$\therefore Omd = 200 \text{ feet.}$$

Eq. (16) gives

$$\frac{S}{2} - AO = \frac{S}{2} \operatorname{vers} \frac{O}{4} = .088.$$

$$\therefore AO = \frac{S}{2} - .088 = 99.912.$$

Eq. (27) gives

$$AK = \frac{S}{2} \cdot \frac{\sin \frac{O}{2}}{3} = 100 \frac{\sin 4^\circ 48'}{3} = \frac{8.368}{3} = 2.789.$$

Or AK and AO may be taken directly from Tables XV and XVI.

TO LAY OUT THE CURVE.

Use four 50-feet chords, for example. $dOg = \frac{O}{3} = 3° 12' = 192'$.

Set the transit at O and turn off from AO.

$$\frac{192'}{16} = 12' \text{ for station } (a);$$

$$12' \times 4 = 48' \text{ for station } (b);$$

$$12' \times 9 = 1° 48' \text{ for station } (c);$$

and $$12' \times 16 = 3° 12' = \frac{O}{3} \text{ for station } (d).$$

Or, having computed the first deflection angle (12′), find it in column 1 of Table XVII, and opposite it, in columns 2, 3, etc., are the deflection angles for stations 2, 3, etc.

Proceed in precisely the same way in any case. That is, divide $dOg = \frac{O}{3}$ by the square of the number of equal chords to be used, which gives the deflection angle for station 1; then multiply this angle by $2^2 = 4$, $3^2 = 9$, etc., for succeeding stations. Or take the angles, after the first, from Table XVII.

Measure the chords Oa, ab, etc., as for a uniform curve.

Example 2.—Let the degree of Kd be $D' = 8° 20'$, and $S = 300$ feet. Then

$$Kd = \frac{300}{2} = 150 \text{ feet}, \quad \text{and} \quad dEK = \frac{150}{100} \times 8° 20' = 12° 30' = O.$$

$$\frac{O}{2} = 6° 15', \quad \frac{O}{3} = 4° 10' = 250', \quad \text{and} \quad \frac{O}{4} = 3° 7\tfrac{1}{2}'.$$

Compute as above, or Table XV gives

$$AK = 5.44,$$

and Table XVI gives $$AO = 149.78.$$

Using chords of 50 feet, the first deflection angle $= \frac{250}{36} = 06'.95$ or 07′, and the succeeding angles to the nearest minute are 28′, 1° 03′, 1° 51′, 2° 54′, and 4° 10′.

It is not necessary to know AK in order to lay out the curve; but AO must be known to give the beginning of the curve.

Problem 2.—Given the degree of the offset curve Kd and the offset AK, to find the length of the transition curve, etc.

Example.—Let radius $EK = R' = 698.73$, deg. $D' = 8° 12' = 492'$. Offset $AK = 4$ feet.

From (37),

$$\text{vers } \tfrac{2}{3}O = \frac{4AK}{3R'} = \frac{16}{2096.19} = .00763;$$

$$\therefore\ O = 10° \ 37\tfrac{1}{2}', \quad \frac{O}{3} = 3° \ 32\tfrac{1}{2}' = 212\tfrac{1}{2}', \quad \text{and} \quad \frac{O}{4} = 2° \ 39\tfrac{1}{2}'.$$

Then

$$Kd = \tfrac{1}{2}S = 100\frac{O}{D'} = 100 \,.\, \frac{10° \ 37\tfrac{1}{2}'}{8° \ 12'} = 100\frac{637.5}{492} = 129.60.$$

Table XVI gives $d = .14$, and therefore $AO = 129.46$.

Use five chords, each $\dfrac{129.6 \times 2}{5} = 51.8$ feet in length.

The first deflection angle is $\dfrac{212\frac{1}{2}}{25} = 8.5$, and the successive angles are 34′, 1° 16½′, 2° 16′, and 3° 32½′.

In case it is desired to use an offset of an approximate given length, it is generally sufficient and best to take S in round numbers so as to correspond with the desired offset sufficiently near, and then proceed as in Problem 1. Table XV very much facilitates this operation. Thus for $D' = 8° \ 12'$ and $S = 250$, we find (by interpolating for the .02′ above 8° 10′) offset = 3.72 feet.

Also,

$$O = \frac{125}{100} D' = 10° \ 15', \quad \frac{O}{3} = 205', \quad \text{and} \quad \frac{O}{4} = 2° \ 34'.$$

Then Table XVI gives $d = .12$, and therefore

$$AO = \frac{S}{2} - d = 124.88.$$

Problem 3.—Given the degree (D') of Kd and the tangent distance AO, to find the length of the transition curve Omd, and the offset AK.

Example.—Let $D' = 9° 36'$, $\therefore R' = 596.83$, andlet $AO = 99.912$. From (11),

$$\sin \tfrac{1}{4}O = 99.912 \div 1342.87 = .07440;$$

$$\therefore \quad O = 9° 36'.$$

From (10),

$$\frac{S}{2} = 99.912 \div .99912 = 100;$$

$$\therefore \quad S = 200.$$

Now (27) or Table XV gives

$$AK = 2.789.$$

Remark.—If S(or $Kd = \frac{S}{2}$) and $AO = \frac{S}{2}\cos\frac{O}{4}$ are given, we have $\cos\frac{O}{4} = AO \div \frac{S}{2}$, which gives O. Then $D' = \frac{O}{Kd}$.

Or, under S in Table XVI, find $d = \frac{S}{2} - AO$, opposite which, in column 1, is D'. Then

$$O = \frac{S}{200}D'.$$

Then $AK = \frac{S}{6}\sin\frac{O}{2}$; or, with D' and S, Table XV gives AK.

If S and $AK = \frac{S}{6}\sin\frac{O}{2}$ are given, $\sin\frac{O}{2} = AK \div \frac{S}{6}$. Or under S in Table XV find AK, opposite which, in column 1, is D'. Then as before.

If AO and AK are given, (31) gives

$$\sin\frac{O}{4} = \frac{3}{2}\frac{AK}{AO}.$$

'10) gives $\frac{S}{2} = AO \sec\frac{O}{4}$. Then as above.

Problem 4.—Given the degree D of the main curve Af, and the length of Kd or of Omd to find the offset AK, the tangent distance AO, etc.

Example 1.—Let $AE = R = 599.62$, $\therefore\ D = 9°\ 33\frac{1}{8}'$. Let $S = 200$.

Then $Kd = 100$, $Af > 100$, and $O = \text{degree of } Kd = D' > D$.

Assume $O = D = 9°\ 33\frac{1}{8}'$. Then $\dfrac{O}{2} = 4°\ 46\frac{9}{16}'$,

and, from (27), $AK = 100 \sin \dfrac{4°\ 46\frac{9}{16}'}{3} = 2.776$.

Then $R' = R - t = 596.844$.

$\therefore\ D' = 9°\ 36' = O_1$; $\dfrac{O_1}{2} = 4°\ 48'$; $\dfrac{O_1}{3} = 192'$, etc.

Now $AO = 100 \cos 2°\ 24' = 99.912$.

Or these values of AK and AO may be taken from the tables.

The above value of offset is only approximate; but AO and not AK, is used in laying out the curve. To show the trifling effect upon Kd and AO of an error in the offset we now compute corrected offset.

We have $AK = t' = 100 \sin \dfrac{4°\ 48'}{3} = 2.789$.

Let R'' and D'' represent the new values of R' and D'.

Then $R'' = R - t' = 596.831$, and $D'' = 9°\ 36' = D'$.

Or Table XV gives opposite $9°\ 33\frac{1}{8}'$ $t = 2.776$.

Then $R' = R - 2.776 = 596.844$, and $D' = 9°\ 36'$.

Now opposite $9°\ 36'$ we find $t' = 2.789$, etc.

It is seen that t', the corrected value of t, exceeds t by only .013, and therefore R'', the corrected value of R', falls short of R' by the same amount; so that D'' exceeds D' by only $\frac{3}{4}$ of a second and does not appear in the result. This $\frac{3}{4}$ of a second in D' or O decreases AO by less than two units of the fifth decimal place.

We thus see that the assumption, *for the purpose of computing the offset*, that the degree of Kd is the same as that of Af, leads to only a small error in the offset and to an exceedingly small error in the essential quantities involved; so that it is useless to recompute the offset, etc.

We also have, from (32),

$$t' = t + \frac{t^2}{R - 2t}.$$

Example 2.—Let $AE = R = 674.07$, $\therefore D = 8° 30'$.

Let $S = 300$, then $Kd = 150$. $Af > 150$, and $O > \frac{3}{2}D$.

Assume $O = \frac{3}{2}D = 12° 45'$, and $\frac{O}{2} = 6° 22\frac{1}{2}'$.

Then $AK = t = \frac{S}{6} \sin \frac{O}{2} = \frac{100 \sin 6° 22\frac{1}{2}'}{2} = 5.55.$

Hence $R' = 674.07 - 5.55 = 668.52.$

$$\therefore D' = 8° 34'; \quad O = \tfrac{3}{2}D' = 12° 51'; \quad \frac{O}{2} = 6° 25\tfrac{1}{2}';$$

$$\frac{O}{3} = 257', \quad \text{and} \quad \frac{O}{4} = 3° 12\tfrac{3}{4}'.$$

It is useless, as already explained, to compute a corrected offset t', since it would not increase D' but little over $2''$, nor decrease AO by a unit of the fourth decimal place.

Now $AO = 150 \cos 3° 12\frac{3}{4}' = 149.76.$

Problem 5.—Given the degree of the main curve and the length of the part Af replaced, to find the offset AK, the tangent distance AO, etc.

Example.—Let $AE = R = 599.62$, $\therefore D = 9° 33\frac{1}{3}'$. Let $Af = 100$.

Then

$$O = D; \ \tfrac{2}{3}O = 6° 22'\tfrac{2}{9}; \ \frac{O}{3} = 191', \ \text{and} \ \frac{O}{4} = 2° 23'\tfrac{1}{3}.$$

By (38),

$$AK = \frac{R \text{ vers } \frac{1}{2}O}{\frac{4}{3} + \text{vers } \frac{1}{2}O} = \frac{599.62 \times .0061746}{1.3395} = 2.764.$$

Hence

$$EK = R' = 599.62 - 2.764 = 596.87,$$

and $$D' = 9° 36'.$$

Now $$\frac{S}{2} = Kd = 100\frac{D}{D'} = 99.54.$$

Table XVI gives for $D' = 9°36'$ and $S = 200$,

$$\frac{S}{2} - AO = .09;$$

$$\therefore\ AO = 99.45.$$

Use four chords each $\frac{99.54}{2} = 49.77$ feet long.

The deflection angle for the first station is $\frac{191}{16} = 12'$, nearly, and others are $48'$, $107\frac{1}{2}'$, and $191'$.

Problem 6.—To replace each half of a simple curve AfA', Fig. 131, of radius R and degree D by a transition curve.

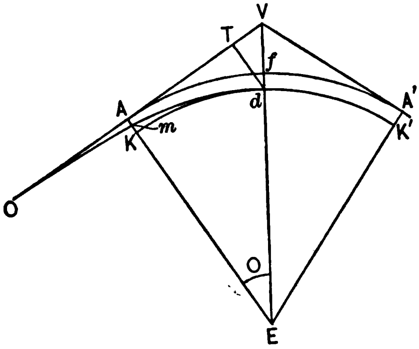


Fig. 131.

Example.—Let $R = 716.2$, $\therefore\ D = 8°$, $AEV = \frac{1}{2}V = O = 19°$, etc.

Eq. (38) gives

$$AK = \frac{716.20 \times .024338}{1.3577} = 12.84.$$

Hence

$$R' = 716.20 - 12.84 = 703.36, \quad \text{and} \quad D' = 8° \, 8\tfrac{3}{4}' = 8°.146.$$

Also, $$\frac{S}{2} = 100\frac{19}{8.146} = 233.24.$$

Table XVI gives $$AO = 232.44.$$

Using eight chords each $\frac{233.24}{8} = 58.3$ feet long, the deflection angles are

$$\frac{6° \, 20'}{64} = 5'.94.$$

$$4 \times 5'.94 = 24';$$

$$9 \times 5'.94 = 53\tfrac{1}{2}';$$

$$16 \times 5.94 = 1° \, 35', \text{ etc.}$$

This problem is the same in principle as the preceding.

The next problem is preferable to it in most cases.

Problem 7.—To connect the tangents AV and $A'V$ (Fig. 131) by two transition curves each of a given length $Od = S$.

Example.—Let $Od = S = 600$, and $AEV = O = 25° \, 36'$.

Now $Kd = \frac{S}{2} = 300.$

$$\text{Degree of } Kd = D' = \frac{100}{Kd}O = \tfrac{1}{3}O = 8° \, 32' = 512'.$$

$$\therefore \; R' = 671.43.$$

Eq. (10) or Table XVI gives

$$AO = 300 \cos 6° \, 24' = 298.13$$

Also, $$AV = R \tan O \qquad = 332.31$$

$$\therefore \; VO = AO + AV \quad = 630.44$$

This gives the tangent point O. Using ten chords each equal to 60 feet, the deflection angle for the first station is $\frac{512}{100} = 5'.12$, and the successive deflection angles are found as usual. AK may be computed as usual, and then $AE = R = AK + R'$; but they are not needed.

Problem 8.—To connect the tangents AV and $A'V$ (Fig. 131) by two equal transition curves having a common vertex distance $dV = e$.

Example.—Let $e = 60$, $AEV = O = 20°$, $\frac{2O}{3} = 13° 20'$, $\frac{O}{3} = 400'$.

From (18), $$\frac{S}{2} = dV \frac{\cos O}{\sin \frac{2}{3}O} = 244.48.$$

From (6),

$$OV = OT + TV = S \cos \tfrac{201}{450}O + dV \sin O = 503.55.$$

This gives the tangent point O.

Using ten chords each $\frac{244.48}{5} = 48.9$ feet long, the deflection angle for the first station from O is $\frac{400'}{100} = 4'$, and the following are found as usual.

This problem serves to connect two tangents by two equal transition curves having a vertex distance equal, for example, to that of any simple curve.

Problems 7 and 8 show that the transition curve may take the place of a simple or compound curve without interfering in the least with its use at the same time as a transition curve. It is therefore more general in its use than those curves.

It may be impracticable to lay out the whole curve from O. Suppose the part $Oc = S$ (Fig. 132) to be laid out from O, and the transit moved to c. Let e represent any station, and $ce = S_1$. Let $Oa = 1$, and θ = the angle turned between O and a. Prolong Oc to any point l. Draw tangents at c and e. Let c and c_1 represent the chords Oc and ce.

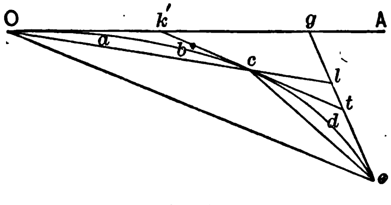


FIG. 132.

Then, see eqs. (45), (46), (47),

$$cOe = S_1(2S + S_1)\frac{\theta}{3};$$

$$ceO = S(2S + S_1)\frac{\theta}{3};$$

and
$$ecl = (S + S_1)(2S + S_1)\frac{\theta}{3}.$$

Now the transit being at c, sight to O, reverse on l, and turn off successive angles found by putting $S_1 = 1$, $S_1 = 2$, etc., in the last equation. These angles are

$$(S + 1)(2S + 1)\frac{\theta}{3}, \qquad (S + 2)(2S + 2)\frac{\theta}{3}, \text{ etc.}$$

Then measure cd, de, etc., as usual.

The curve may be easily retraced from e toward O.

We have
$$tcl = OcK' = 2AOc = 2S^2\frac{\theta}{3}.$$

$$\therefore\ ect = ecl - tcl = [(S + S_1)(2S + S_1) - 2S^2]\frac{\theta}{3} = S_1(3S + S_1)\frac{\theta}{3}. \quad (50)$$

Again,

$$cet = Oet - Oec = 2AOe - ceO$$

$$= 2(S + S_1)^2\frac{\theta}{3} - S(2S + S_1)\frac{\theta}{3} = S_1(3S + 2S_1)\frac{\theta}{3}. \quad . \quad (51)$$

Let $S' = S + S_1$. Then, in (51),

for the first station from e put $S = S' - 1$, and $S_1 = 1$;
for the second station from e put $S = S' - 2$, and $S_1 = 2$, etc.,

giving the angles $(S' - \frac{1}{3})\theta$, $(S' - \frac{2}{3})2\theta$, etc.

Turn off these angles in succession from the tangent et, and locate d, c, etc.; otherwise as usual.

Problem 9.—To substitute a transition curve for an end portion of a main curve without moving the rest of the curve.

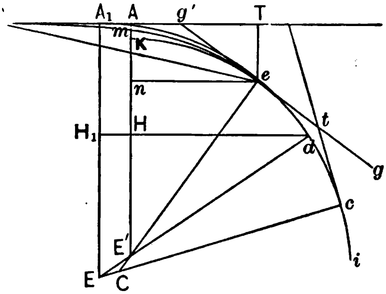


FIG. 133.

Let A_1dc (Fig. 133) be a part of the main curve tangent to OT at A_1. Suppose the curve deK, with $dE' = R' < dE = R$, to be run from some point d of the curve A_1dc, through an angle $KE'd = A_1Ed = O_1$, and draw dHH_1 parallel to OT.

Suppose a transition curve emO, having a central angle $KE'e = O$, to connect this curve with the tangent OT at O. The point d must be taken far enough from A_1 to give room for the transition curve between d and the tangent AO; and R' cannot be $< \frac{3}{4}R$, for, if so, KH would be $< \frac{3}{4}AH$, or AK would be $> \frac{1}{4}AH$, and no transition curve could connect Ked and the tangent AO.

Draw en parallel to OT. Then

$$Kn = \tfrac{3}{4}An.$$

In general we have

$$AH = A_1H_1 = R \text{ vers } O_1, \quad HK = R' \text{ vers } O_1.$$

$$\therefore \frac{KH}{AH} = \frac{R'}{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

If $R' = \frac{3}{4}R$, $KH = \frac{3}{4}AH$ and the transition curve begins at d.

If $R' > \frac{3}{4}R$, $KH > \frac{3}{4}AH$, or $AK < \frac{1}{4}AH$, and the transition curve begins at some point between K and d.

We have, from the above,

$$AK = AH - KH = (R - R') \text{ vers } O_1.$$

Also, $$nK = 3AK = 3(R - R') \text{ vers } O_1.$$

$$\therefore \frac{HK}{nK} = \frac{R'}{3(R - R')}.$$

But
$$\frac{HK}{nK} = \frac{R' \operatorname{vers} O_1}{R' \operatorname{vers} O} = \frac{\operatorname{vers} O_1}{\operatorname{vers} O}.$$

$$\therefore \frac{\operatorname{vers} O_1}{\operatorname{vers} O} = \frac{R'}{3(R - R')},$$

or
$$\operatorname{vers} O_1 = \frac{R'}{3(R - R')} \operatorname{vers} O = \frac{D}{3(D' - D)} \operatorname{vers} O. \quad (53)$$

Since Kd and A_1d subtend the same angle,

$$\frac{Kd}{A_1d} = \frac{R'}{R} = \frac{D}{D'}.$$

We have seen that if from any point d of a curve A_1dc of radius R a curve deK be described with radius $R' = \frac{3}{4}R$, then a transition curve connecting A_1dc with the tangent AO_1 will begin at d.

When, however, A_1dc is a sharp curve, the change from radius R to radius $\frac{3}{4}R$ would be rather too sudden, and therefore it is best to take $R' > \frac{3}{4}R$ and commence the curve at a corresponding point e given by eq. (53).

$$R' = \tfrac{5}{6}R, \quad \text{or} \quad R' = \tfrac{7}{8}R, \quad \text{or} \quad R' = \tfrac{9}{10}R$$

are practical values for R'.

The ratio $\frac{HK}{nK} = \frac{R'}{3(R - R')}$ shows that the larger R' is the larger $\frac{HK}{nK}$ is, and therefore the longer the connecting branch de.

It is not necessary to run the curve deK much beyond e; but if run to K, we thus determine A, which is opposite to K. In this case the position of T is not needed.

A short solution of this problem sufficiently accurate, unless O is *very* large, is the following:

We have approximately

$$\frac{\overline{Kd}^2}{\overline{Ke}^2} = \frac{HK}{nK}, \quad \text{or} \quad Kd = Ke\left(\frac{HK}{nK}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (54)$$

This formula gives only slightly too small a value for Kd.

Example 1.—Let $R = 954.9$ ($D = 6°$), and $Ome = 200$.

Let $R' = \frac{9}{10}R$, then $D' = \frac{10}{9}D = 6° \ 40'$. To find d, e, and O, and to locate the curves.

We have $$KE'e = O = D'\frac{Ke}{100} = D' = 6° \ 40';$$

$$\text{vers } O_1 = \frac{R'}{3(R - R')} \text{ vers } O = 3 \text{ vers } O = .02028;$$

$$\therefore \quad O_1 = 11° \ 56'.$$

Hence $$A_1d = 100\frac{11.56}{6} = 192.7;$$

$$Kd = \frac{R'}{R}A_1d = \frac{9}{10}A_1d = 173.4;$$

$$de = Kd - Ke = 73.4.$$

Or, by (54), $$Kd = 100\sqrt{3} = 173.2;$$

$$de = Kd - Ke = 73.2;$$

$$A_1d = \frac{10}{9} \times 173.2 = 192.4, \text{ etc.}$$

Now $$AO = \frac{S}{2} \cos \frac{O}{4} = 100 \cos 1° \ 40' = 99.96,$$

or $$TO = S \cos \tfrac{1}{2}O = 200 \cos 3° \ 18' = 199.70.$$

Also, $$\frac{O}{3} = 133'.$$

For four equal chords the first deflection angle is

$$\frac{133'}{16} = 8'.3.$$

Other deflection angles are found as heretofore.

Measure off $AO = 99.96$ feet or $TO = 199.70$ feet; set the transit at O. and locate the transition curve *Ome* as usual.

Then observing that $Oeg' = \frac{2}{3}O$, giving the position of the tangent at e, locate the branch de.

Example 2.—Given $R = 716.2$ $(D = 8°)$, $R' = \frac{8}{9}R$, $\therefore D' = \frac{9}{8}D = 9°$, and $S = 300$ feet, to find the tangent points d, e and O, and to locate the curves.

Since $Ke = \dfrac{S}{2} = 150$ feet,

$$KE'e = \tfrac{3}{2}D' = 13° \, 30' = O.$$

$$\text{vers } O_1 = \frac{R'}{3(R - R')} \text{ vers } O = \tfrac{8}{3} \text{ vers } O = .07368.$$

$$\therefore \quad O_1 = 22° \, 08' = 22°.13.$$

Then $$Kd = 100\frac{22.13}{9} = 245.9,$$

and $de = Kd - Ke = 95.9$ ft.; $A_1d = \frac{9}{8}Kd = 276.6$.
Or, first find A_1d, then Kd and de as in the preceding example.

Now $$AO = \frac{S}{2}\cos\frac{O}{4} = \frac{300 \cos 3° \, 22'}{2} = 149.74.$$

Also, $$\frac{O}{3} = 270'.$$

For six equal chords the first deflection angle $= \dfrac{270'}{36} = 7'.5$.
Locate the transition curve *Ome*, and then the branch *de* as shown above.

If $KE'e = O$ is given instead of S, we have

$$Ke = \frac{S}{2} = 100\frac{O}{D_1},$$

which gives S. Then find O_1 as before, etc.

If it is desired to make a given offset *t*, we may find *S* by Table XV, corresponding to *t* and *D'*, and then proceed as before.

COMPARISON BETWEEN THE TRUE TRANSITION CURVE AND THE COMPOUND CURVE COMMONLY USED FOR A TRANSITION CURVE.

This compound curve is one of any number of branches, the curve being "compounded at the end of each chord."

The degree of the first branch is made the common difference of the degrees of the curve of the succeeding arcs.

Thus suppose the chords each 100 feet long, and the degree of the first branch 10′. Then the degrees of the succeeding arcs are 20′, 30′, etc.

Such a curve inclines to the tangent, at all points, too much, the inclination at the middle of any chord being precisely what it should be at the end of the chord.

This has the effect of making the tangent distance somewhat too short and the offset much too long.

Thus at the end of the tenth chord the approximate results are: tangent distance = 997.32 and offset = 55.89. Whereas the true tangent distance = 998.82 and offset = 48.40.

The errors increase with the sharpness of the curve and with its length.

It is to be noted that the true transition curve is continuous; whereas the approximate transition curves, used as such, are discontinuous. Any point on the former can be determined at once and without reference to other points; whereas, in the latter, points on the curve must be found *consecutively*. To find a point at the extremity of a chord, for example, points at the extremities of all preceding chords must be found.

Thus the laying out of such a curve is, for this among other reasons referred to, necessarily laborious.

Mr. E. W. Crellin, C.E., now of Des Moines, Iowa, explained the nature and to some extent the properties of the true transition curve in an article in the *Transit*, published April, 1890, by the State University of Iowa.

The author is not aware, however, that any general discussion of the curve has heretofore been made; that its general properties have been found; that practical formulas for it have been deduced; and that its applicability has been shown.

A writer on transition curves says that "the theoretic curve which should be adopted is a form of the elastic curve, which, on account of the trouble in locating it, has been supplanted by various approximations, such as the curve of sines, parabolas, etc.; these being easier to locate in the field."

The true transition curve, as we have seen, is, on the contrary, much more easily located than any of the approximate curves used.

CHAPTER X.

CALCULATION OF EARTHWORK.

THIS chapter, for obvious reasons, does not constitute a complete treatise on earthwork.

The object here sought is to treat in the most simple manner, and by means of the shortest formulas, the usual cases that arise in practice.

To compute the contents of an excavation it is, in general, necessary to divide it into a number of prismoids by cross-sections at the stations and at other suitable points.

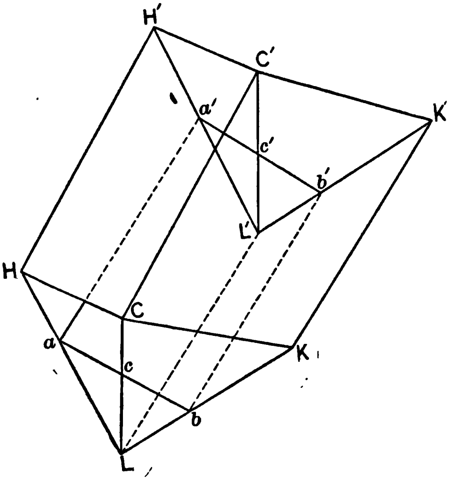


FIG. 134.

A prismoid is a volume the ends or bases of which are parallel planes, and the lateral surfaces are either planes or warped surfaces, such as may be generated by a right line moving along two lateral edges as directrices, these edges being right lines joining corresponding angles or corners in the bases.

Thus let Fig. 134 represent a prismoidal excavation between the cross-sections at C and C'.

$HabKC$ represents one end, and the same letters accented represent the other end. $aba'b'$ represents the roadbed, $aa'HH'$ and $bb'KK'$ represent the side slopes. $CC'HH'$ represents the surface of the ground on the left of the center line CC', and $CC'KK'$ the same on the right.

FORMULAS FOR SECTIONAL AREAS.

Let b = the width of roadbed;
S = slope of sides;
c = depth at center stake;
h and k = depth at side stakes;
d and e = distances from center to side stakes.

A. For ground level laterally the section $abHK$ (see Fig. 135) is a trapezoid, having an average width of

$$Aa + ab = cS + b, \quad \text{and} \quad \text{area} = a = c(cS + b). \quad . \quad (1)$$

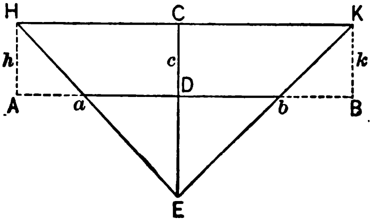


FIG. 135.

B. When the ground slopes are not the same on opposite sides of the center line as shown in Fig. 136, we find as follows:

Prolong the side slopes Ha and Kb to intersect in E.

Now $$DE = \frac{\frac{1}{2}b}{S} = \frac{b}{2S}.$$

Then $$\text{area } CHEK = \tfrac{1}{2}CE \times AB = \frac{1}{2}\left(c + \frac{b}{2S}\right)(d + e);$$

$$\text{area of } abE = aD \times DE = \frac{b^2}{4S}.$$

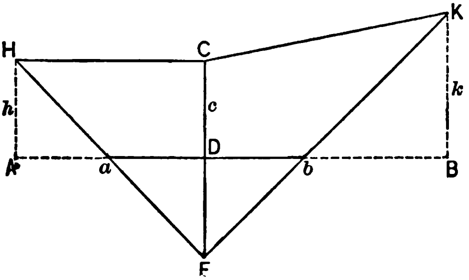


FIG. 136.

Then $$\text{area } CHabk = a = \frac{1}{2}\left(c + \frac{b}{2S}\right)(d + e) - \frac{b^2}{4S}.$$

We observe that $\frac{b}{2S}$ and $\frac{b^2}{4S}$ are constant for the same slope and width of roadbed. Represent the former by i and the latter by a' and we have

$$\text{area } abHK = a = \tfrac{1}{2}(c + i_1)(d + e) - a'. \quad . \quad . \quad (2)$$

For another section

$$a_1 = \tfrac{1}{2}(c_1 + i)(d_1 + e_1) - a'. \quad . \quad . \quad . \quad (3)$$

Let $CE = c + i = C$, and $AB = d + e = D$, and we have

$$\text{area of } CHabK = a = \tfrac{1}{2}CD - a'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly for another section

$$a_1 = \tfrac{1}{2}C_1D_1 - a'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These are the simplest formulas possible for this case, since each requires but a single multiplication, while any other formula requires two multiplications. These formulas have been used by the author for more than thirty years. See correspondence in the *Engineering News*, vol. 32, page 73.

When the surface of the ground is irregular between the center and side stakes it is best to divide the cross-section into trapezoids by dropping perpendiculars to the roadbed at those stakes, and at each *break* in the surface between them, thus forming a number of trapezoids as shown in Fig. 137. Then the area of the section is equal to the area of the trapezoids less the area of the triangle *bBK*.

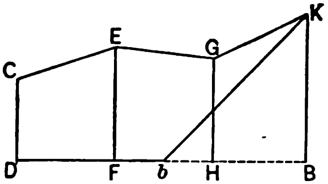


FIG. 137.

FORMULAS FOR REGULAR EXCAVATIONS AND EMBANKMENTS.

A. The volume of a prismoid may be exactly calculated by the prismoidal formula,* which is

$$V = \frac{l}{6}(a + 4m + a_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which V = volume, l = length ; a, a_1 = the areas at the ends, and m = the area of the middle section.

To find the volume in cubic yards, divide the above by 27.

Since the linear dimensions of the middle section are arithmetical means of the corresponding dimensions of the end sections, we have

$$m = \frac{1}{2}\left(\frac{C + C_1}{2}\right)\left(\frac{D + D_1}{2}\right) - a' = \tfrac{1}{8}(C + C_1)(D + D_1) - a', \quad (7)$$

in which a' = the area of the triangle under the roadbed.

---

* The late Prof. W. M. Gillespie proved, many years ago, that the prismoidal formula is applicable to a prismoid having one warped surface; and the late Prof. De Volson Wood afterwards proved that the same formula is applicable to a prismoid having two warped surfaces.

The author proved (in 1869) that the formula in question applies directly to a prismoid having any number of lateral surfaces, any number or all of which are warped.

Mr. A. M. Bannister, C.E., of Lansing, Mich., Mr. Cornelius Donovan, C.E., of Port Eads, La., and others are acquainted with the facts. An elegant proof was since given by Prof. George Bruce Halsted of Austin, Texas.

B. Substituting the values of a, a_1, and m in (6), we find

$$\text{vol} = l\left(\frac{a}{2} + \frac{a_1}{2}\right) - \frac{l}{12}(C - C_1)(D - D_1); \quad . \quad (8)$$

$$\text{vol} = lm + \frac{l}{24}(C - C_1)(D - D_1). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We observe that $C - C_1 = c - c_1$.

$l\left(\frac{a}{2} + \frac{a_1}{2}\right)$ is called the "end area volume," and lm the middle area volume. The above equations show that the former differs from the true volume twice as much as the latter and in the opposite sense. This may be shown symbolically as follows:

$$\frac{l}{6}(a + 4m - a_1) - \frac{l}{2}(a + a_1) = \frac{l}{3}(2m - a - a_1);$$

$$\frac{l}{6}(a + 4m + a_1) - lm = \frac{l}{6}(a + a_1 - 2m).$$

The last of these equations is one half of the first with a contrary sign.

In case the prismoid is level laterally we have, from Fig. 135,

$$D = AB = ab + 2Aa = b + 2cS.$$

Similarly $D_1 \qquad = b + 2c_1S.$

$$\therefore \; D - D_1 = 2S(c - c_1).$$

Hence $$\text{vol} = l\left(\frac{a}{2} + \frac{a_1}{2}\right) - \frac{lS}{6}(c - c_1)^2, \quad . \quad . \quad . \quad (10)$$

or $$\text{vol} = lm + \frac{lS}{12}(c - c_1)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

To generalize equation (8):

Let $a, a_1, a_2 \ldots a_n$ represent areas of sections;
$c, c_1, c_2 \ldots c_n$ represent center heights;
and $d, d_1, d_2 \ldots d_n$ represent the extreme width of sections, as AB, Fig. 136,

Then expressing volumes of successive prismoids, and adding, we get

$$\text{vol} = l(\tfrac{1}{2}a + a_1 + a_2 + \ldots + \tfrac{1}{2}a_n)$$
$$- \frac{l}{12}[(c - c_1)(d - d_1) \ldots (c_{n-1} - c_n)(d_{n-1} - d_n)]. \quad (12)$$

The areas of one half of the end sections appear, and the whole areas of other sections.

The correction for each prismoid must be taken separately.

Let Fig. 138 represent a longitudinal section of a "cut" through the center line. Drop the perpendiculars CD, C_1D_1, etc., and at the middle points of DD_1, D_1D_2, etc., erect perpendiculars ME_1, M_1F, etc., to meet horizontal lines through C, C_1, etc.

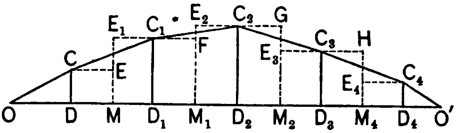


FIG. 138.

Then the first part of formula (12) gives the volume *represented* by $CDME + E_1MM_1F + E_2M_1M_2G +$ etc., and the second part furnishes the required corrections.

Example.—Let $b = 20$ and $s = 1$. Then (see Fig. 136) $DE = 10$ and area $abE = 100$. Commencing at C, let the center heights be 3.0, 6.0, 7.0, 5.0, and 2.5, and distances out be 26.0, 33.2, 34.8, 31.6, and 26.0.

Now, by (2) or (4),

$$\tfrac{1}{2}a = \tfrac{1}{4} \times 13 \times 26 - 50 \quad = \ 34.5$$
$$a_1 = \tfrac{1}{2} \times 16 \times 33.2 - 100 = 165.6$$
$$a_2 = \tfrac{1}{2} \times 17 \times 34.8 - 100 = 195.8$$
$$a_3 = \tfrac{1}{2} \times 15 \times 31.6 - 100 = 137.0$$
$$\tfrac{1}{2}a_4 = \tfrac{1}{4} \times 12.5 \times 26 - 50 \ = \ \underline{31.25}$$

Then $564.15 \times 100 = 56415.$

$$
\begin{aligned}
\text{Also,}\quad (c - c_1)(d - d_1) &= 3.0 \times 7.2 = 21.6 \\
(c_1 - c_2)(d_1 - d_2) &= 1.0 \times 1.6 = 1.6 \\
(c_2 - c_3)(d_2 - d_3) &= 2.0 \times 3.2 = 6.4 \\
(c_3 - c_4)(d_3 - d_4) &= 2.5 \times 5.6 = \underline{14.0} \\
& \qquad\qquad\quad 43.6 \times \tfrac{100}{12} = 363.
\end{aligned}
$$

Therefore volume = 56052 cu. feet.

These quantities may be taken at once from suitable tables, thus requiring only a few additions to give volumes of about as many prismoids.

The prismoids ending at O and O' are best computed separately.

To compute CDO.

Let $DO = l' = 75$. The distance out at $O = b = 20$.

$$
\begin{aligned}
\text{Then}\qquad \tfrac{1}{2}al' &= 34.5 \times 75 = 2587.5 \\
\text{and}\qquad \tfrac{75}{12}(c - 0)(d - b) &= 6.25 \times 3 \times 6 = \underline{112.5} \\
\therefore\ \text{volume} &= 2475.0
\end{aligned}
$$

If the ground is not level laterally in the region of O (or O'), we must take cross-sections where the center line or either edge of the roadbed comes to grade and compute the volumes between the cross-sections separately.

Generalizing eq. (10), we have

$$
\begin{aligned}
\text{vol} = l(\tfrac{1}{2}a + a_1 + \ldots + a_{n-1} + \tfrac{1}{2}a_n) \\
- \frac{ls}{6}[(c - c_1)^2 + (c_1 - c_2)^2 + \ldots (c_{n-1} - c_n)^2]. \quad (13)
\end{aligned}
$$

The areas may be computed by eq. (4), and the quantities within the brackets by a table of squares; or all may be found by suitable earthwork tables.

SPECIAL FORMULAS AND CASES.

Let $CDK \ldots C'$, Fig. 139, represent a triangular prismoid.

Let $CD = c$, $C'D' = c_1$, $KB = K$, $K'B' = K_1$, $DB = d$, and $D'B' = d_1$.

Let a and a_1 represent the areas of CKD and $C'K'D'$; m = the area of the middle section; l = the length.

We readily find

$$\text{vol.} = \frac{l}{6}[\tfrac{1}{2}cd + \tfrac{1}{2}c_1d_1 + \tfrac{1}{2}(c + c_1)(d + d_1)] = \frac{l}{6}(a + a_1 + 4m), \quad (14)$$

$$= lm + \frac{l}{24}(c - c_1)(d - d_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14')$$

$$= \frac{l}{4}\left[(cd + c_1d_1) - \frac{l}{12}(c - c_1)(d - d_1)\right]$$

$$= l\left(\frac{a}{2} + \frac{a_1}{2}\right) - \frac{l}{12}(c - c_1)(d - d_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14'')$$

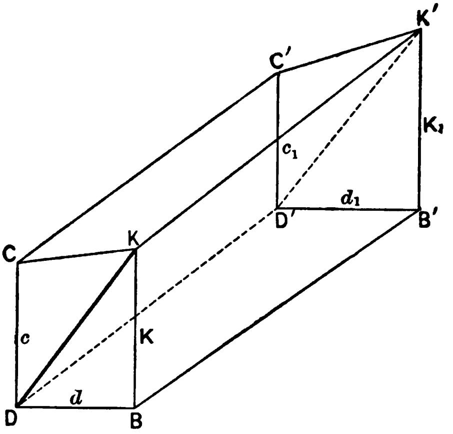


FIG. 139.

These equations will evidently give the volumes on both sides of the center line by letting d and d_1 represent the sum of the distances out at the respective ends of the prismoid.

The above equations also give the volume of $KBD \ldots K'$ by substituting K and K' in place of c and c'.

Hence

$$\text{vol } KBD \ldots K' = \frac{l}{6}[\tfrac{1}{2}Kd + \tfrac{1}{2}K_1d_1 + \tfrac{1}{2}(K + K_1)(d + d_1)]$$

$$= \frac{l}{6}(a + a_1 + 4m), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$= lm + \frac{l}{24}(K - K_1)(d - d_1), \quad . \quad . \quad . \quad (15')$$

$$= \frac{l}{4}(dK + d_1K_1) - \frac{l}{12}(d - d_1)(K - K_1)$$

$$= l\left(\frac{a}{2} + \frac{a_1}{2}\right) - \frac{l}{12}(d - d_1)(K - K_1). \quad (15'')$$

APPLICATION OF GROUP (15).

If $d = d_1$ and $K = K_1$, the prismoid becomes a prism and any equation of the group gives

$$\text{vol} = l\frac{Kd}{2} = la.$$

If $d_1 = K_1 = 0$, the figure is a pyramid and vol $= \frac{lKd}{6} = \frac{la}{3}$.

If $\frac{d}{d_1} = \frac{K}{K_1}$ or $dK_1 = d_1K$, the prismoid becomes a frustum of a pyramid and

$$\text{vol} = \frac{l}{6}(dK + d_1K_1 + d_1K),$$

or

$$\text{vol} = \frac{l}{6}(dK + d_1K_1 + dK_1).$$

The formula given in the books for the volume of a frustum of a pyramid is

$$\text{vol} = \frac{l}{3}(a + \sqrt{aa_1} + a_1).$$

The formula above is much more simple.

FORMULA FOR THE VOLUME OF ANY FRUSTUM OF A PYRAMID.

Let d and K represent any two sides of one base, and d_1 and K_1 corresponding sides of the other base.

Let $a = ndK$, then

$$a_1 = nd_1K_1, \quad \text{and} \quad aa_1 = n^2dKd_1K_1;$$

or putting $d_1K = dK_1$, we have

$$aa_1 = n^2d^2K_1^2, \quad \text{and} \quad \sqrt{aa_1} = ndK_1.$$

Hence $\frac{l}{3}(a + \sqrt{aa_1} + a_1) = \frac{ln}{3}(dK + d_1K_1 + dK_1)$.

In this substitute $n = \frac{a}{dK}$ or $n = \frac{a_1}{d_1K_1}$.

In the above $n = \frac{1}{2}$, and $\therefore$ vol $= \frac{l}{6}(dK + d_1K_1 + d_1K)$.

Adding (14″) and (15″), we have

$$\left.\begin{array}{l} \text{vol} = \frac{l}{4}[d(c+K)+d_1(c_1+K_1)] - \frac{l}{12}(d-d_1)(c-c_1+K-K_1), \\ \text{or} \\ \text{vol} = l\left(\frac{a}{2}+\frac{a_1}{2}\right) - \frac{l}{12}(d-d_1)(c-c_1+K-K_1), \end{array}\right\} \tag{16}$$

since $d(c + K) = 2a$, and $d_1(c_1 + K_1) = 2a_1$.

This applies to a prismoid having vertical sides and a trapezoidal base.

If $d = d_1$, the base is a rectangle, and we have, from the above,

$$\text{vol} = \frac{ld}{4}(c + K + c_1 + K_1) = l\left(\frac{a}{2} + \frac{a_1}{2}\right). \quad . \quad . \tag{17}$$

For n prismoids of common length l and width d we have

$$\left.\begin{array}{l} \text{vol} = \frac{ld}{4}[(c+K)+2(c_1+K_1)+\ldots+2(c_{n-1}+K_{n-1})+(c_n+K_n)]; \\ \text{vol} = l\left(\frac{a}{2}+a_1+a_2+\ldots+a_{n-1}+\frac{1}{2}a_n\right). \end{array}\right\} \tag{18}$$

If $d_1 = 0$, the edges at C' and K' coincide, $c_1 = K_1$, and we have, from (16),

$$\text{vol} = \frac{ld}{2} \cdot \frac{c + K + c_1}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \tag{19}$$

The figure is a wedge, $\frac{ld}{2}$ being the area of the base BDD', and $\frac{c + K + c_1}{3}$ the average height of the edges.

If $c = K$, and $c_1 = K_1$, in Fig. 139, (16) becomes

$$\text{vol} = \frac{l}{2}(cd + c_1d_1) - \frac{l}{6}(c - c_1)(d - d_1). \quad . \quad . \quad (20)$$

We observe with reference to all prismoids that the bases may occupy any relative position in the parallel planes without in the least affecting their volumes.

LOADED FLAT CARS, PILES OF STONE, ETC.

Eq. (20) gives the volume of a flat car loaded with earth, gravel, etc., or a pile of stone, etc., c and d being the dimension of either base, and c_1 and d_1 those of the other base.

Usually the side slopes in such cases are nearly uniform, but the formula applies, as we have seen, if each slope is uniform, whether they are equal or not, and also when the side slopes are warped surfaces.

The volume of a borrow-pit divided into prisms having equal or equivalent bases, of area $= A$, is readily found by a single calculation.

Let Fig. 140 represent the base of such an excavation, the heights at the corners being denoted by the letters there placed. The total volume is equal to $\frac{1}{4}A$ multiplied by the sum total of the sum of the corner heights of the several prisms. Into this total sum the corner heights a, a_1, b, b_3, d, d_3 will enter but once, being found in but one prism; c, c_3, d_1, and d_2 will enter twice, being common to two prisms; b_1 and b_2 will enter three times, being common to three prisms. Finally, c_1 and c_2 will enter four times, being common to four prisms.

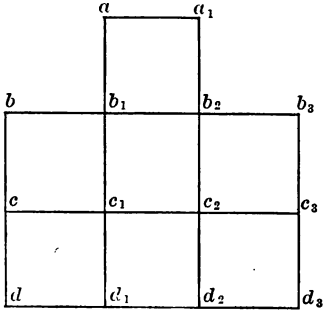


FIG. 140.

Representing these sums in order by S_1, S_2, S_3, and S_4, we have, for all the prisms,

$$\text{vol} = \tfrac{1}{4}A(S_1 + S_2 + S_3 + S_4). \quad . \quad . \quad . \quad . \qquad (21)$$

ENDS OF EMBANKMENTS OR "DUMPS."

Let Fig. 141 represent an embankment.

Let c and c_1 represent the center heights and d and d_1 the horizontal widths of HK and H_1K_1.

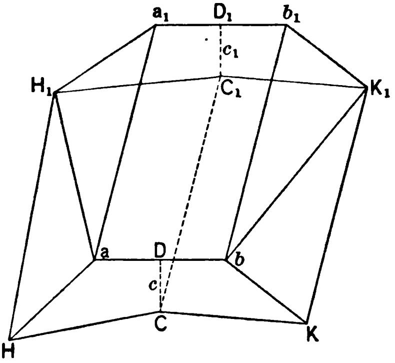


FIG. 141.

Draw aH_1 and bK_1. Then, by (8),

$$\text{vol } abHCK \ldots H_1C_1K_1 = l\frac{a}{2} - \frac{l}{12}c(d - d_1), \quad . \qquad (22)$$

since $\qquad c_1 = 0, \quad \text{and} \quad a_1 = 0.$

If $d = d_1$,

$$\text{vol of dump} = l\frac{a}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

GROUND IRREGULAR LATERALLY.

Let Fig. 142 represent the part of a prismoid on the right of the center line CC' for the case supposed, which requires the intermediate heights at D, E, D', and E'. The total volume is equal to the volume of the prismoids whose bases are $mFHn$, $FGLH$, and $GBB'L$, less the volume of the sub-slope prismoid $bBK \ldots b'B'K'$.

In general the volumes of the former are found by (16). When, however, the widths at the two ends of the prismoid are equal, (17) applies.

When more heights are taken at one end than at the other, the width of one or more prismoids at one end is 0 and (19) applies.

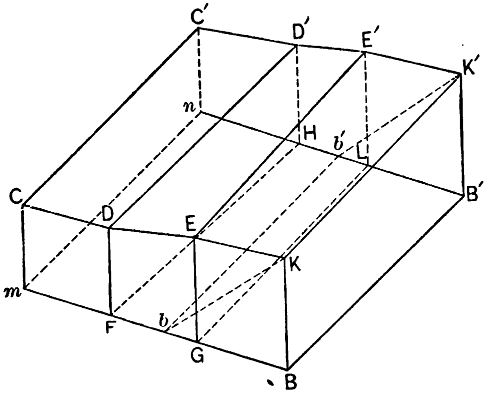


FIG. 142.

The volume of the sub-slope prismoid is found by any equation of group (15).

MIXED WORK, EXCAVATION AND EMBANKMENT.

Suppose, for example, there is excavation on both sides and embankment on one side only.

Let Fig. 143 represent the cross-section at one end of the pris-

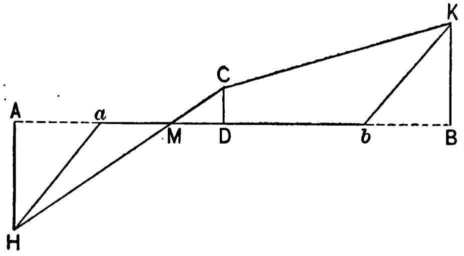


FIG. 143.

moid. *CDbK* shows the cut on the right, *CDM* that on the left, and *aHM* the fill on the left. *Db* is the half-width of roadway in the cut, and *aD* the half-width in the fill and cut. *HCK* shows the surface of the ground, and *aH* and *bK* the side slopes.

The volume of the cut on the right is found at once by (8), observing that in this case a and a_1 represent the areas on one side only of the center line, and D and D_1 the distances out on one side only.

The volume of the cut on the left is found by (14), by substituting the values of *CD* and *DM* for c and d, and making similar substitutions for c_1 and d_1 at the other end of the prismoid.

The volume of the embankment is found by means of the same equations by substituting the values of *aM* and *AH* in place of c and d, etc.

In case a fill is on both sides of the center line and a cut on one side only, the same formulas of course apply.

A cross-section must be taken at the point where the surface of the ground intersects the center of the roadbed. At this point the triangle *CDM* vanishes, *CDbK* becomes *DbK*, and *HaM* becomes *HaD*.

Then the volume of the cut, of which *CDbK* shows one end and *DbK* may represent the other, is given by (8), in which $c_1 = 0$.

The volume of the fill, of which *aHM* shows one end and *aHD* may represent the other, is found by (14), as before.

The volume of the pyramid, of which *CDM* shows one end and the point D represents the apex, is given by an equation following group (15), and is

$$\text{vol} = \frac{lKd}{6},$$

in which $K = CD$, and $d = DM$.

The prismoids adjacent to the cross-section where the change considered occurs, but on the opposite side of it to those just considered, are, it is evident, computed in the same way and by means of the same formulas.

A cross-section must be taken, also, where the surface of the ground intersects the side of the roadbed. In this case the same formulas apply to the prismoids on either side of the section in question, and adjacent to it, in precisely the same way as in the case just considered.

CORRECTION OF EARTHWORK FOR CURVATURE.

Suppose the center line of the prismoid (Fig. 139) is an arc of a circle of radius R.

Let $e =$ the length, measured on the line CC', of a very short prismoid, at CDK or $C'D'K'$, or at the middle section.

Then the lengths measured through the centers of gravity of these prismoids are as follows:

The length at CDK is........ $e\frac{R + \frac{1}{3}d}{R}$;

The length at $C'D'K'$ is...... $e\frac{R + \frac{1}{3}d_1}{R}$,

and that at the middle section is $e\frac{R + \frac{1}{3}dm}{R}$.

It is evident, however, that we may consider the lengths through the centers of gravity of the elementary prismoids to be the same as their lengths along CC' if we, at the same time, increase the areas of the middle and of the end sections accordingly.

We must write, therefore, $\frac{R + \frac{1}{3}d}{R}A$ in place of A,

$\frac{R + \frac{1}{3}d_1}{R}A_1$ in place of A_1,

and $\frac{R + \frac{1}{3}dm}{R}m$ in place of m.

Hence

$$\text{true vol} = \frac{l}{6}\left(\frac{R + \frac{1}{3}d}{R}A + \frac{R + \frac{1}{3}d_1}{R}A + 4\frac{R + \frac{1}{3}dm}{R}m\right), \quad (25)$$

and

$$\text{the excess of vol} = \frac{l}{6}\left(\frac{\frac{1}{3}d}{R}A + \frac{\frac{1}{3}d_1}{R}A_1 + 4\frac{\frac{1}{3}dm}{R}m\right), \quad (26)$$

$$= \frac{l}{18R}(dA + d_1A_1 + 4dm \times m). \quad (27)$$

OVERHAUL.

No allowance is made for moving excavated material when the haul does not exceed a certain specified distance called the "limit of free haul." But when the material is carried beyond this limit the extra labor involved is paid for at a stipulated price per cubic yard, per each 100 feet in *excess* of the free-haul limit.

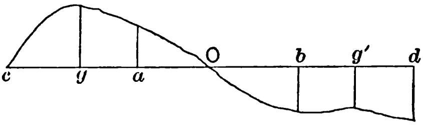


FIG. 144.

Let us suppose the material in the cut *cO*, Fig. 144, just sufficient to make the fill *Od*.

First find on the profile two points *a* and *b* such that the cut *aO* will just make the fill *Ob*, and that the distance *ab* is equal to the limit of free haul.

These points *a* and *b* are found by means of the cross-sections and calculated quantities; though since *a*, as well as *b*, usually falls between regular stations, it is generally necessary to find the point *a* by one or more trials, so that *ab* is equal to the required limit.

Having found *ab*, it is to be remembered that the contractor is entitled to pay for moving every cubic yard of material from the cut *ca* to the fill *bd* for the whole distance it is moved, less the distance *ab*.

This is equivalent to moving the whole cut a distance $gg' - ab = ga + bg'$; g and g' being respectively the centers of gravity of the cut *ca* and of the fill *bd* made from it.

But the volume of *ac* multiplied by the distance *ag* is equal to the sum of the products obtained by multiplying the volume of each prismoid in *ac* by the distance of its own center of gravity from *a*.

It is usually sufficiently accurate to consider the center of gravity of a prismoid as being at its mid-section; but if greater accuracy is required, we have for the distance from the mid-section

$$x = \frac{l}{6} \cdot \frac{A - A_1}{A + A_1}, \text{ very nearly. } \quad . \ . \ . \ . \quad (28)$$

This is, of course, to be added to or subtracted from the distance of the mid-section from the point *a* accordingly as the larger end area *A* is the farthest from or nearest to the point *a*.

In the same manner we find the sum of the products obtained by multiplying the volume of each prismoid in *bd* by the distance of its center of gravity from *b*.

Summing the products for both cut and fill, the distances being expressed in chains of 100 feet, and multiplying the result by the stipulated allowance, we have the amount to be paid for haul.

Parts of a cut may be carried in opposite directions, in which case each part must be figured separately.

It is evident that no allowance is made on material wasted.

MONTHLY ESTIMATES.

Monthly estimates are usually made by the resident engineer near the end of each month, though it is sometimes necessary to make them at other times. For this purpose calculated quantities in the field-books are used, supplemented by such measurements as may be necessary.

The monthly estimate is only approximate, and should not be above the real amount of work done; but it should be as definite and complete as the nature of the case will permit, and should include a detailed statement of all the work done and material delivered.

A special field-book is devoted to monthly estimates. Cross-sections must be taken for the purpose of computing the amount of excavation completed; and notes of everything done must be made.

Where work is completed, the corresponding quantities may be taken from the field-books or other books containing them.

An allowance somewhat below the actual value should be made for all material delivered but not yet used, as well as for all labor performed and expenditures made in properly forwarding the work.

Estimate sheets should be used, the sheet for each month showing, for different parts of the work and for each kind of material, the total of the previous estimates, the present estimate, and the total including the present estimate.

The total estimate for any month becomes the total "previous estimate" for the succeeding month.

The division engineer, or the chief engineer in case there is no division engineer, reviews these reports, copies them on other sheets; attaches the prices to the items; computes and sums up the amounts.

The railway company pays the contractor, each month, about 85% of the estimate for that month, and retains the rest till the completion of the contract.

FINAL ESTIMATE.

The final estimate is a complete statement in detail of all the work done and of all the material furnished by the contractor, and furnishes the basis of final settlement between the company and the contractor.

This statement is completed, in detail, as the work progresses. As soon as the data of any part or subdivision of the work, or of any structure, are supplied, a complete statement in regard to it should be written out in detail in a book for that purpose. The number of cubic yards in each prismoid, extra for cutting surface ditches, overhaul, etc., should be given. A complete statement in regard to each bridge or other structure should be carefully written out. The notes should be made especially full, while the work is in progress, in regard to all parts of it which are inaccessible after completion, such as foundation-pits and foundations of all kinds, and all works under water.

COMPUTATION OF PRISMOIDS—LEVEL LATERALLY.

Let *abmn* (Fig. 145) represent one end of an embankment, 100 feet in length, level laterally.

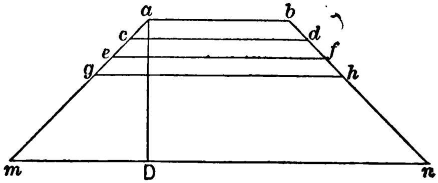


FIG. 115.

Let *cd*, *ef*, etc., represent horizontal planes which divide the 'bankment into layers 0.1 of a foot in height.

Let the base $ab = b$, and the side slope $\frac{mD}{aD} = s$. Then

$$ab = b;$$
$$cd = b + 0.2s;$$
$$ef = b + 0.4s, \text{ etc.}$$

Also, $\frac{ab + cd}{2} = b + 0.1s$, $\frac{cd + ef}{2} = b + 0.3s$, etc.

Hence

the vol. in yards, of $abcd = \frac{0.1 \times 100}{27}(b + 0.1s) = \frac{10b + s}{27}$,

and vol of $cdef = \frac{10}{27}(b + 0.3s) = \frac{10b + 3s}{27}$, etc.

It is plain that the volumes of the successive layers are in arithmetical progression, having a common difference of $\frac{2s}{27}$. Hence to find the total volumes corresponding to different heights of the embankment, differing by 0.1, it is only necessary to write down $\frac{10b + s}{27}$ for the volume of $abcd$; add $\frac{10b + 3s}{27}$ to it for the volume of $abef$, then $\frac{10b + 5s}{27}$ to the last result for the volume of $abgh$, etc., omitting, for convenience, the denominator 27.

Example 1.—Let $b = 18$, and $s = 1$. Then

$$\frac{10b + s}{27} = 6 + \tfrac{19}{27}, \quad \frac{10b + 3s}{27} = 6 + \tfrac{21}{27}, \quad \text{etc.}$$

Hence we have as follows:

Heights.	Volumes.		Heights.	Volumes.	
0.1	6 6	19 21	0.4	27 7	7 0
0.2	13 6	13 23	0.5	34 7	7 2
0.3	20 6	9 25	0.6	41 etc., etc.	9

This may easily be carried 20 or 30 feet in an hour, or less.

Example 2.—Let $b = 14$, and $s = \frac{3}{2}$. Then

$$\frac{10b + s}{27} = 5 + \tfrac{13}{54}, \quad \frac{10b + 3s}{27} = 5 + \tfrac{19}{54}, \quad \text{etc.}$$

Then as follows :

Heights.	Volumes.		Heights.	Volumes.	
0.1	5 5	13 19	0.5	27 5	17 43
0.2	10 5	32 25	0.6	33 5	6 49
0.3	16 5	3 31	0.7	39 6	1 1
0.4	21 5	34 37	0.8	45 etc.,	2 etc.

These results can be copied to the nearest yard very quickly.

The author hopes to show the best processes of computing all earthwork and other tables, and of making all numerical computations, in the near future.

CHAPTER XI.

APPROXIMATE AND ABRIDGED COMPUTATIONS.

THIS chapter will treat of the subject of approximate and abridged computations to a very limited extent only, dealing mainly, too, with the practical side of the question.

Approximate computations are those that lead to results not strictly accurate.

Abridged computations are those that reach certain results with less labor than the ordinary computations involve. Abridged computations may be either accurate or approximate, but are usually the latter. The absolute error of a result is the difference between that result and the true result for which it stands.

The relative error of a result is the absolute error divided by the true result, and it is therefore equal to the absolute error of each unit of the result.

Thus if a true result is 84 and the approximate value 77 is taken for it, the absolute error is $84 - 77 = 7$, and the absolute error of each unit of the result, or the relative error, is $\frac{7}{84} = \frac{1}{12}$.

It is important to notice that the relative error of a result is not changed by multiplying that result by any number; since the error in the product or quotient will be increased or decreased in the same ratio as the result is increased or decreased.

Thus if n be the true and $n + e$ the approximate result, then $r = \frac{e}{n}$ is the relative error. Multiplying by m, we have the approximate product $m(n + e) = mn + me$, whereas the true product is mn. Hence the absolute error of the product is me, and the relative error is $\frac{me}{mn} = \frac{e}{n}$ as before.

If an approximate quantity is given, the absolute error of which is a given number of units of a given order counting from the highest, it follows from the preceding that the relative error is independent of the decimal point. Thus suppose the absolute error of each of the numbers 7.073, .07073, and 707.3 to be two units of the fourth order. The left-hand significant figure expresses units of the first order, and therefore the fourth order is expressed in the above by the figure 3. The relative errors of the numbers are $\frac{.002}{7.073}$, $\frac{.00002}{.07073}$, and $\frac{.2}{707.3}$, each of which is equal to $\frac{2}{7073}$.

Hence if a number is given whose absolute error does not exceed a given number of units of any order (the nth, say), an approximate and convenient limit of the relative error is readily found by dropping all the figures at the right of the nth figure, replacing with zeros all the others, except one or two at the left, and dividing the given number of units by the result, regarded as a whole number. Thus we know that $\pi = 3.1415$ to within less than a unit of the fifth order or fourth decimal place, and hence the relative error cannot exceed $\frac{.0001}{3.1415} = \frac{1}{31415}$, or $\frac{1}{30000}$, or $\frac{1}{31000}$, etc.

The first n highest figures of an approximate number express the value of that number to n places. The same figures, increasing the last by unity, when the following figure is 5 or more, express the *exact* value to n places of the same number. Thus 3.141592 being an approximate value of π, 3.1415 is called the value of π to five places, and 3.1416 is called the *exact* value of π to five places.

ADDITION.

To find the relative error of the sum of any number of approximate quantities.

Let $n_1, n_2 \ldots n_m$ represent the true values of the quantities, and $e_1, e_2 \ldots e_m$ represent their absolute errors. Let S represent the sum of the quantities, and E the sum of the absolute errors.

Then

$$E = e_1 + e_2 + \ldots e_m, \quad S = n_1 + n_2 + \ldots n_m,$$

and the relative error is

$$r = \frac{E}{S} = \frac{e_1 + e_2 + \ldots e_m}{n_1 + n_2 + \ldots n_m}. \quad . \quad . \quad . \quad . \quad (1)$$

Supposing the errors to be on the same same side, then, in case they are equal, the relative error of the sum of the quantities is equal to the relative error of each of the quantities; but when they are unequal, this error of the sum is greater than the smallest relative error of the quantities, and smaller than the greatest of such errors. If the errors are on opposite sides of the truth, the relative error of the sum must be less than the greatest relative error, and may be equal to zero.

It is sufficient, therefore, in practice to take each number to be added as accurate as it is desired that the result shall be.

Example.—Find the sum of the square roots of the numbers 2, 3, 5, 6, 7, 8, 10, 11, 12, and 13 true to about the fifth place. Taking the square roots exact to five places, we have the following result:

$$\begin{array}{rl}
1.4142 & + \\
1.7321 & - \\
2.2361 & - \\
2.4495 & - \\
2.6458 & - \\
2.8284 & + \\
3.1623 & - \\
3.3166 & + \\
3.4641 & + \\
3.6056 & - \\
\hline
26.8547 & \\
\hline\hline
\end{array}$$

Without examination we know from the preceding that the relative error of this result must be less than $\frac{\frac{1}{2}}{14142} = \frac{1}{28284}$, which is less than $\frac{1}{26854}$. Hence the absolute error of the result must be less than a unit of the fifth place, or less than .001. Since, however, there are only ten numbers added, the error in the right-hand or sixth figure of the result cannot exceed $\frac{1}{2} \times 10 = 5$, and the error in the fifth figure cannot exceed $5 \div 10 = \frac{1}{2}$. Furthermore, there are only six errors in the same direction, and hence the error in the fifth place cannot exceed $\frac{1}{2} \times 6 \div 10 = \frac{3}{10}$ of a unit.

SUBTRACTION.

Let n_1 and n_2 represent two quantities, n_1 being greater than n_2; and let $n_1 + e_1$ and $n_2 + e_2$ represent approximate values of the same. The true difference is $n_1 - n_2$; the difference between the approximate values is

$$n_1 + e_1 - (n_2 \pm e_2) = (n_1 - n_2) + e_1 \mp e_2.$$

Hence $e_1 \pm e_2$ is the absolute error of the difference due to errors in the numbers.

The relative error of the difference is

$$\frac{e_1 \mp e_2}{n_1 - n_2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The minuend and subtrahend are usually taken true to $\frac{1}{2}$ of a unit of the last place, in which case the absolute error of the difference cannot exceed a unit of the same place or order of units.

The relative error of two numbers, nearly equal, may be quite large, as shown in the following example:

Compute $\sqrt[3]{83} - \sqrt{18}$ correct to five places and find the relative error of the result.

$$\sqrt[3]{83} = 4.3621 \text{ to within } \tfrac{1}{2} \text{ a unit of the fifth place.}$$

$$\sqrt{18} = 4.2426 \text{ to within } \tfrac{1}{2} \text{ a unit of the fifth place.}$$

$$\text{Difference} = 0.1195 \text{ to within 1 unit of the fourth place.}$$

The relative error of either of the numbers cannot exceed $\dfrac{\frac{1}{2}}{42426} = \dfrac{1}{84852}$, while the relative error of the difference may be nearly $\dfrac{1}{1195}$, which is 71 times as large as the former.

Any degree of accuracy may be secured in the result or difference, however, by taking the numbers sufficiently accurate.

MULTIPLICATION AND DIVISION.

To find the error in a product due to errors in one or both factors.

Let n_1 and n_2 represent any numbers,, $n_1 + e_1$ and $n_2 + e_2$ approximate values of the same as above.

Let $N = n_1 n_2$, $N_1 = (n_1 + e_1)(n_2 + e_2)$, and let $E = N_1 - N =$ the absolute error of the product.

Now $N_1 - N = n_1 e_2 + n_2 e_1 + e_1 e_2 = n_1 e_2 + n_2 e_1$, very nearly,

and therefore

$$\frac{N_1 - N}{N} = \frac{E}{N} = \frac{n_1 e_2 + n_2 e_1}{n_1 n_2} = \frac{e_1}{n_1} + \frac{e_2}{n_2}, \quad . \quad . \quad . \quad (3)$$

or, generally,

$$\frac{E}{N} = \frac{e_1}{n_1} \pm \frac{e_2}{n_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The two results given by eq. (4) occur equally often. Hence the relative error of the product of two numbers can never exceed the sum of the relative errors of the numbers, and it is as often equal to the difference of such errors as to their sum.

If $e_2 = 0$,

$$\frac{E}{N} = \frac{e_1}{n_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence if only one factor is in error, the relative error of the product is equal to the relative error of the factor in error.

If e_1 and e_2 are in opposite directions, and $\frac{e_1}{n_1} = \frac{e_2}{n_2}$, then

$$\frac{E}{N} = \frac{e_1}{n_1} - \frac{e_2}{n_2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence two numbers may be changed in opposite directions and in proportion to their magnitudes without sensibly affecting their product. Thus $41 \times 82 = 3362$, and $40 \times 84 = 3360$, Comparing multiplication with division, we know that the product in the former corresponds to the dividend in the latter, and the factors in the former to the divisor and the quotient in the latter. Hence $\frac{E}{N}$ may represent the relative error of the dividend, and $\frac{e_1}{n_1}$ and $\frac{e_2}{n_2}$ such errors in the divisor and the quotient. Transposing eq. (4), therefore, and we have

$$\frac{e_1}{n_1} = \frac{E}{N} \mp \frac{e_2}{n_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This shows that the relative error of the quotient (or divisor) is equal to the difference, or to the sum, of the relative errors of the dividend and divisor (or quotient).

It follows that when the divisor is correct the relative error of the quotient is equal to that of the dividend, and when the dividend is correct the relative error of the quotient is equal to that of the divisor, but with a contrary sign.

Furthermore, the divisor and the dividend may be both increased or both decreased, in proportion to their magnitudes, without affecting the quotient.

If $n_1, n_2 \ldots n_m$ represent factors, and N their product, etc., then

$$\frac{E}{N} = \frac{e_1}{n_1} + \frac{e_2}{n_2} + \ldots \frac{e_m}{n_m}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

e_1, e_2, etc., may be either positive or negative, and hence the relative error of the product is equal to the algebraic sum of the relative errors of the factors.

Without here discussing the average value of the relative error of the product, it is sufficient to state that it rarely much exceeds the greatest relative error in the factors ; and hence the factors only need be taken to about that degree of accuracy desired in the product. Similarly, the divisor and the dividend should be taken to about that degree of accuracy desired in the quotient.

Example.—Find the product 89.0245 × .0194525 true to about four places.

89.02 × .01945 = 1.731439, true to less than ½ a unit of the fourth place, Suppose we have, for example, $N = abc + def + gh$, and suppose the relative errors in the factors are each equal to e.

Then it may be shown that the relative error of the product cannot exceed the relative error of that term, which contains as many factors as any. In the expression above, the relative error of N could not exceed the relative error of the term abc or def, or three times the common relative error of the factors, or $3e$.

Suppose $Q = \dfrac{abc + de}{kl + m}$. Then the relative error of Q cannot exceed the greatest relative error of any term in the numerator, plus the greatest relative error of any term in the denominator; or the relative error of a term equal to the product of such terms. These terms in the above are abc and kl, and hence the relative error in Q cannot exceed that of $abckl$, or $5e$.

Since the errors of the factors are usually quite unequal and on opposite sides of the truth, the relative error of the product or of the quotient rarely much exceeds the greatest relative error of the factors; and hence in all cases it is practically sufficient to take the factors to about that degree of accuracy desired in the result.

ABRIDGED MULTIPLICATION.

To find the product of two factors true to a unit of the nth order.

Rule.—Write the multiplier in the inverse order, placing the highest order under the nth order of the multiplicand. Multiply by each figure of the multiplier, beginning with the figure in the multiplicand above it, rejecting the part of the multiplicand to the right of it, except to carry from, and place the right-hand figures of the partial products under each other. Annex at the right of the partial product a number of zeros, equal to the number of figures in the multiplier and multiplicand less $n + 1$, and point off from the right for decimal factors as usual.

With respect to products containing decimals, the decimal point is easily determined by inspection, and in this case the zeros need not be added.

Examples.—Find 615694 × 59019, 367847 × 278437, and 3.141592 × 1.4142, probably true to a u of the nth or fourth order.

Arranging the factors and multiplying as directed, we find

615694	367847	3.141592
91095	734872	24 141
30785	7357	3 142
5541	2575	1 257
6	294	31
6	15	12
	1	1
36338000000	102420000000	4.443

With respect to the above, we observe that, usually, as in the first and second examples, the right-hand figures of the partial product are of the $n + 1$ (fifth in this case) order, and that the error in each of these figures, if the carrying has been attended to, cannot exceed $\frac{1}{2}$ of a unit, or $2\frac{1}{2}$ units in all, or $2\frac{1}{2} \div 10 = \frac{1}{4}$ of a unit of the fourth order. For reasons already given, the error will rarely exceed 1 unit of the $n + 1$ order.

When the highest figures of the factors are sufficiently small, as in the third example, so that there are only n figures in any of the partial products, and also in their sum, the right-hand figures of the partial products will be of the nth order, and it is possible for the result to be in error more than a unit of the nth order, though such an error is quite improbable.

In this case we observe, also, that the left-hand figures of the products are always larger than those of either factor, so that a possible error of over a unit in the nth place of the result is not of much practical consequence. To make *sure* of avoiding this possible error, the right-hand figure of the inverted multiplier must, in this case, be placed under the $n + 1$ figure of the multiplicand.

To find the product of any number of factors true to about a unit of the nth place, proceed with each product by the above rule.

Example.—Find the product, true to about a unit of the fourth place, of 762,8314 × 6,821426 × 4827.31 × .027265.

762.83	5203.6	25119000
4128 6	37284	56272
4577 0	20814	50238
610 2	4163	17583
15 3	104	502
8	36	151
3	2	13
5203.6	25119000	684870

The right-hand figures of the multipliers are placed under the *n*th (fourth) figure of the multiplicand, except in finding the last product, where it is placed under the $n + 1$ (fifth), so that there will be $n + 1$ figures in each product. The number of integral figures in each product is easily found by inspection.

DIVISION.

A divisor and a dividend being given to any required degree of exactness, to find the quotient true to about a unit of the *n*th order.

Rule.—Take the divisor to n significant figures and enough exact figures in the dividend (n or $n + 1$) to contain this divisor. Multiply the divisor by the first quotient figure, and subtract the product from the assumed dividend; and instead of annexing any figure to the remainder, reject the right-hand figure of the divisor to determine the second quotient figure. Continue the process of division by rejecting successively the right-hand figures of the divisor until the quotient contains n figures.

In multiplying by the first quotient figure carry from the part of the divisor at the right of the *n*th figure, and in each multiplication carry from the part of the divisor at the right of the part multiplied, so as to make each partial product true, if possible, to within ½ a unit of the lowest order.

Example.—Find the quotient of 1497.82746 divided by 16.72374 to within about a unit of the fifth order. Place the divisor on the right of the dividend, and the quotient under the divisor, and cross out, when used, the successive right-hand figures of the divisor. Thus we have

$$
\begin{array}{llr}
 & & 149783)1\cancel{6}\cancel{7}\cancel{2}\cancel{3}74 \\
 & & 133790\ \overline{89563} \\
\cline{3-3}
\text{1st remainder} & & 15993 \\
 & & 15051 \\
\cline{3-3}
\text{2d} & \text{“} & 942 \\
 & & 836 \\
\cline{3-3}
\text{3d} & \text{“} & 106 \\
 & & 100 \\
\cline{3-3}
(n-1)\text{th} & \text{“} & 6 \\
 & & 5 \\
\cline{3-3}
n\text{th} & \text{“} & 1
\end{array}
$$

The last partial product (5 in the example) and the last remainder (1 in the example) need not in practice be written.

To determine the possible error in the quotient, we will find the possible error in the last remainder used in finding the quotient.

Now the error in the dividend cannot exceed $\frac{1}{2}$ a unit of the lowest order, and the error in each partial product cannot exceed $\frac{1}{2}$ a unit of its lowest order, which is the same as that of the dividend. The error in the $(n-1)$th remainder (6 in the example) cannot exceed the sum of these errors, or

$$\tfrac{1}{2} + \tfrac{1}{2} \times (n-1) = \frac{n}{2}.$$

The nth or last quotient figure (3 in the example) is found by dividing the $(n-1)$th or next to the last remainder (6 in the example) by the first figure in the divisor (having regard to carrying from the other figures), and cannot exceed that remainder. But there is no error in the divisor, and hence the relative error of the last quotient figure (and hence of the quotient) cannot exceed the relative error of next to the last remainder; and since the last quotient figure cannot exceed the next to the lowest remainder, the absolute error of the quotient cannot exceed the absolute error of that remainder, or $\frac{n}{2}$.

It is plain that the error in question is usually less than unity.

Example 1.—Find the quotient $\frac{194169}{763054}$ with an absolute error in the quotient less than a unit of the sixth place.

$$\begin{array}{rl}
1941690) & \underline{7\dot{6}\dot{3}\dot{0}\dot{5}\dot{4}} \\
1526108 & 254463 \\ \hline
415582 & \\
381527 & \\ \hline
34055 & \\
30522 & \\ \hline
3533 & \\
3052 & \\ \hline
481 & \\
458 & \\ \hline
23 &
\end{array}$$

Example 2.—Compute the expression $\frac{\pi}{\sqrt{2}}$ with a relative error of less than .0001.

We easily see that the quotient will be greater than 2, and hence we may make an absolute error of .0002. It will be more than sufficient, then, to find the quotient true to a unit of the fourth decimal place.

Thus we find

$$\begin{array}{rl}
3.1415) & \underline{1.\dot{4}\dot{1}\dot{4}\dot{2}} \\
28284 & 2.2214 \\ \hline
3131 & \\
2828 & \\ \hline
303 & \\
283 & \\ \hline
20 & \\
14 & \\ \hline
6 &
\end{array}$$

CHAPTER XII.

CONSTRUCTION.

CLEARING AND GRUBBING.

THE first operation in constructing a railway is to clear off all timber upon the right of way. The engineer should provide in his specifications for the cutting into proper lengths, and storing, all valuable timber near the borders of the right of way. The refuse should be burned.

Where a deep cut is to be made, the trees are felled and the stumps removed as the earth is excavated, and the stumps are covered up in deep fills. In very shallow cuts and fills it is best to pull over, and out, the trees by their roots, rather than to fell them and afterwards to grub out the stumps, though this is often done.

Since in fills the tops of stumps should be at least 1½ feet below grade, it is easy to tell the trees that can best be cut down, and those that must be grubbed out.

The amount of grubbing required being usually difficult to estimate accurately, it is decidedly the best way to make no separate account of it, but to include it in the earthwork.

GRADE LINE.

Having a profile of the line, a grade line is established thereon, first tentatively by the eye, and then definitely determined by computation, so as to make the cost of the earthwork a minimum, due regard being had to total cost and controlling conditions, such as the height of bridges, road-crossings, etc., and especially drainage.

The natural waterways must be crossed high enough to afford sufficient openings for the flow of the water.

In order to form a correct idea as to the size of openings and culverts needed, the engineer must observe the water-marks on trees, etc., note the width and slope of valleys, and, if need be, measure, approximately, the area of watersheds adjacent to the line. It is perhaps needless to note what every engineer of experience knows, that much property, and some lives, are often lost by neglecting these precautions. It is well to observe, too, that the concentration of water in drainage ditches very much increases the flow; since when it is widely spread out over marshes, meadows, etc., much of it, or all of it, may soak into the soil or evaporate.

The concentration of small streams by means of side ditches, thus avoiding unnecessary culverts, is often economical.

Ample side ditches should be constructed to keep the water off the roadway.

Surface ditches should also be made along the line, a few feet from the upper edge of cuts, to prevent the surface-water from pouring into these after rains. Neglect of this causes much unnecessary labor, and sometimes leads to accidents.

It is very desirable to substitute light fills, in place of cuts, so far as practicable, since in cuts it is difficult to maintain a good track.

Having established the grade line, the difference between its height and that of the surface of the ground at different points gives the cuts or fills at these points, along the center line. These should be plainly marked on the center stakes in feet and tenths. Moreover, the heights of the ground at or near all points between stations, where it changes much, should be found, and the corresponding cut or fill marked on stakes set for the purpose. It is well to observe that cross-sections may be chosen between such points, thus diminishing their number, but giving and taking so as to determine the true volume as accurately as though all the points were used.

A cross-section should be taken not only where the irregularities of the center line require it, but also where the irregularities on either side may require it, the object being to delineate the outline of the earthwork, and also to furnish the data for an accurate computation of the same. A "berm" 5 or 6 feet wide should be left between the edges of all side ditches and the foot of the adjacent fill.

STAKING OUT EARTHWORK.

A. When the ground is level laterally, as shown in Fig. 146, the operation is very simple.

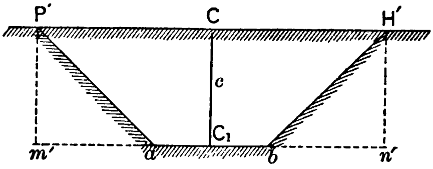


FIG. 146.

Let $2b$ = the width of the roadbed ab, and $c = CC_1$, the center height. The side heights $P'm'$ and $H'n'$ are also equal to c. Let S = the slope of the side $P'a$ or $H'b$. Then

$$bn' = H'n' \times S = cS, \quad \text{and} \quad CH' = d = b + cS.$$

This shows the distance out from C to H', where the excavation must begin, in order that the foot of the slope at b may be at the proper distance $C_1b = b$ from the center C_1.

If in cross-sectioning, whether level laterally or not, the heights of additional points between stations are required, they must be found.

It is also necessary to find the point where the grade line intersects the surface of the ground; since at that point there is neither cut nor fill.

A stake is set there and marked 0.0.; and the corresponding side stakes are set and marked. Also the points where the *edges* of the proposed roadbed intersect the surface of the ground are found, and 0.0. is marked on the stakes set at those points. Cross-sections are taken at these points also.

Let Fig. 147 represent the case in which the ground is not level laterally. Instead of the cross-section $abP'H'$, as in the former case, we now have the cross-section $abPCA$, PCA being the surface of the ground.

We will first show how to find the position of the side stakes by computation, where the slope of the ground is uniform from the center stakes to the side stakes; not for the purpose of actually locating the stakes in this way, for the ground rarely slopes uniformly in any direction, (and therefore this could not be done with much accuracy).

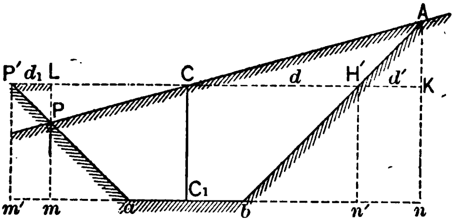


FIG. 147.

It will, however, aid in showing the effect of the slope of the ground in cross-sectioning, as we shall see.

Let $CH' = d$, $H'K = d'$, $CK = d + d' = D$; $S =$ the side slope, and $S' =$ the slope of the ground. Also, $LP' = d_1$, $CL = E$.

Then
$$S = \frac{d'}{AK},$$

and
$$S' = \frac{d + d'}{AK}.$$

Dividing gives
$$\frac{S'}{S} = \frac{d' + d}{d'},$$

from which
$$d' = H'K = \frac{dS}{S' - S} = \frac{S(\frac{1}{2}b + cS)}{S' - S}. \quad . \quad . \quad . \quad (1)$$

Adding $CH' = d$ to the preceding gives
$$CK = D = \frac{dS'}{S' - S} = \frac{S'(\frac{1}{2}b + cS)}{S' - S}. \quad . \quad . \quad . \quad . \quad (2)$$

Dividing (1) by $CH' = d$ gives

$$\frac{d'}{d} = \frac{H'K}{CH'} = \frac{S}{S' - S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Dividing (1) by (2) gives

$$\frac{d'}{D} = \frac{H'K}{CK} = \frac{S}{S'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

These equations enable one to judge quite accurately the *additional* distance beyond H' necessary to go out on the upper side of a cut in consequence of the slope of the ground.

Thus if $S = \frac{S'}{4}$, $d' = \frac{d}{3}$; if $S = \frac{S'}{5}$, $d' = \frac{d}{4}$, etc.

Eq. (3) shows that the additional distance $H'K$ is to the distance CH', for a level section, as S is to $S' - S$, and (4) shows that $H'K$ is to CK as S is to S'.

It is evident that these formulas apply directly to the lower side of an embankment, which Fig. 147 will represent by turning it over.

Example.—Let the roadbed be 20 feet wide, center cut 10 feet, slope of the ground 4 to 1, or $S' = 4$, and side slope 1 to 1, or $S = 1$. Find the distance out to H.

We have $$CK = \frac{4(10 + 10)}{4 - 1} = 26\tfrac{2}{3}.$$

Also, $$H'K = \frac{1(10 + 10)}{4 - 1} = 6\tfrac{2}{3}.$$

$$CH' = 10 + 10 = 20.$$

On the lower side of the cut we have

$$S = \frac{P'L}{PL} = \frac{d_1}{PL}, \quad \text{and} \quad S' = \frac{CL}{PL}.$$

Dividing, we have

$$\frac{S'}{S} = \frac{CL}{d_1} = \frac{d - d_1}{d_1},$$

or

$$d_1 = LP' = \frac{dS}{S' + S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Subtracting this from $CP' = d$, we have

$$CL = CP' - LP' = d - d_1 = E = \frac{dS'}{S' + S} = \frac{S'(\frac{1}{2}b + cS)}{S' + S}. \qquad (6)$$

Dividing (5) by (6), we have

$$\frac{LP'}{CL'} = \frac{S}{S'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Dividing (5) by $CP' = d$, we have

$$\frac{LP'}{CP'} = \frac{S}{S' + S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

These equations show that the subtractive distance LP' is to the distance CP', for a level section, as S is to $S' + S$, and LP' is to CL as S is to S'.

It is evident that these formulas apply at once to the upper side of a fill.

Example.—Let $b = 20$, $c = 10$, $S' = 4$, and $S = 1$.

To find CL, the distance of P from the center C.

We have
$$CL = \frac{4(10 + 10)}{4 + 1} = 16.$$

Also,
$$LP' = \frac{20 \times 1}{4 + 1} = 4.$$

Let Fig. 148 represent a side-hill section.

I. To find by trial the position of H. Let $c = 10.4$ and $b = 1$. It will be convenient to consider the height of the roadbed equal to 0.

Distance out for level section $= d = CH' = CP' = 10 + 10.4 = 20.4$.

Suppose we judge the ground near H' to be 5 feet higher than at C, or 15.4 feet. The distance out corresponding to that height would then be 5 feet beyond H', or

$$10 + 15.4 = 25.4 \text{ feet.}$$

But the ground 5 feet beyond H' is still higher than at H', requiring a still further distance out. Let us test it at e, 26.5 feet out. Suppose we find the ground 6.6 feet higher than at C. This requires a distance out of $20.4 + 6.6 = 27$ feet, or 0.5 of a foot beyond e. But, as before, at 27 feet the ground is a little higher than at 26.5 feet, so we will try it at H, 27.2 feet out, for example.

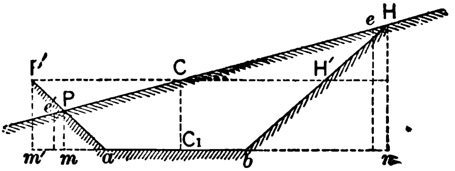


FIG. 148.

Suppose we find this point 0.2 feet above e, or 6.8 above C. This requires a distance out of $27.0 + 0.2 = 27.2$. Hence the height and distance out correspond and are therefore correct.

On the lower side of a fill we would, for reasons already given, proceed in precisely the same way.

II. To find the position of P.

Let us suppose the *ground* near P' to be 6 feet lower than at C. This would require a distance out of $20.4 - 6 = 14.4$ feet.

But 6 feet inside of P' the ground is higher than at P, and, according to the supposition, less than 6 feet below C, and requires, therefore, a distance out greater than 14.4.

Suppose we find the ground at e', 17.4 feet out, 4.8 feet below C, or 5.6 feet. This requires a distance out of $20.4 - 4.8 = 15.6$ feet, or $10 + 5.6 = 15.6$ feet.

But at 15.6 feet out the ground is higher than at e', and therefore the distance out must be greater than 15.6 feet.

Suppose we now find the elevation at P, 16 feet out, to be 0.4 above e', or 6.0 feet. This height and the distance out now correspond and are correct.

We are now prepared to state two important principles which will greatly facilitate the process of cross-sectioning:

I. If, on the upper side of a cut or lower side of a fill, any height found calls for a given change, e, in the distance out, either inward or outward, the real change required is always greater than e, and the excess increases with the lateral steepness of the ground.

Thus, referring to Fig. 149 and the preceding notation, observing that $H''c'$ is the additional distance out (beyond H') corresponding to the additional height $H''H'$ above C, we have

$$\left.\begin{matrix}\text{the real addi-}\\ \text{tional dis-}\\ \text{tance out}\end{matrix}\right\} = H'K = H''c = H''c' \frac{CK}{CH'} = H''c \frac{S'}{S' - S}. \qquad (9)$$

II. If, on the lower side of a cut or upper side of a fill, the height found calls for a given change, e, in the distance out, either inward or outward, the real change required is always less than e, and the deficiency increases with the lateral steepness of the ground.

We have as above, observing that $P''d'$ is the distance to come in corresponding to the fall $P'P''$ below C,

$$\left.\begin{matrix}\text{the real}\\ \text{distance}\\ \text{to come in}\end{matrix}\right\} = P'K' = P''d = P''d' \times \frac{CK'}{CP'} = P''d \frac{S'}{S' + S}. \qquad (10)$$

The author has used these principles for thirty years, but has not seen them stated.

Example.—Let $ab = 24$, $c = 10.5$, $S = \frac{3}{2}$, to find the position of H. (Fig. 149.) $CH' = 12 + \frac{3}{2}(10.5) = 27.8$.

Suppose we judge the ground near H' (at H'') to be 6 feet higher than at C. Then we know that the distance out is greater than $27.8 + 9 = 36.8$. The additional distance corresponding to $H''H'$ is $H''c'$, and $c''c' = 36.8$. Suppose we try it at e, 39.0 feet out and find e 8.6 feet higher than C, or 19.1 feet, calling for a distance of $27.8 + 12.9 = 40.7$ feet, or 1.7 feet beyond e.

We will therefore try it at H, 41.7 feet out, and suppose we find the ground 0.7 feet higher than at e. This would call for an additional foot beyond 40.7, or 41.7 as we have it. The cut is therefore $19.1 + 0.7 = 19.8$, and distance out 41.7 ft.

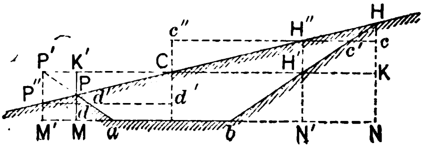


FIG. 149.

To find the position of P.

We have $CP' = 27.8$. Suppose we judge the ground near P' (at P'') to be 6 feet lower than at C; then we know that P is inward from P'' *less* than 9 feet, or $P''d'$, and the distance out is therefore greater than $27.8 - 9.0 = 18.8$ feet $= d'd''$.

Suppose we try it at e', 22.0 feet out, and find e' 5.2 feet lower than C, or 5.3 feet, calling for a distance out of $27.8 - 8.0 = 19.8$ feet, or 2.2 inside of e'. Suppose we come in 1.2 feet or 20.8 feet out to P, and find it 0.6 feet higher than at e' and requiring a distance out of $19.8 + 0.9 = 20.7$. As this is but 0.1 inside of the point of observation, we can probably set P at 20.7 feet out with sufficient accuracy.

It will be seen that the first point of observation is chosen with reference to the known distance CH' for a level section, and that subsequent points are chosen with reference to the figures obtained at the *last* point observed.

A table which can be computed in a few minutes, giving distances out for the given side slope corresponding to different heights, will greatly facilitate the work and likewise conduce to accuracy.

If the ground is irregular laterally, the heights must be found at all points along the cross-section where the slope changes, and these heights, as well as their distances out, must be carefully recorded.

When the surface of the ground intersects the roadbed as vn in Fig. 150, we have what is called side-hill work.

Let *POCH* represent the surface of the ground. *H* is found as already explained, and *O* is found by simply finding a point on *HC* prolonged, whose height is equal to the height of the roadbed or zero. To find *P* we have $\frac{ma}{mP} = S$, and we must find a point *P* on *CO* prolonged that will satisfy this equation. To do this it is only necessary to observe that *ma* increases faster than *mP*; and hence to increase $\frac{ma}{mP}$ we must move outward, and to decrease the same we must move inward.

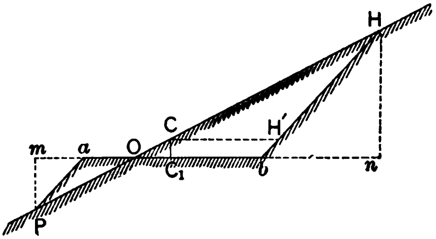


FIG. 150.

When two materials are found in the same section, as rock overlaid with earth, it is necessary to give each material its proper slope.

Since the upper layer is usually of varying thickness, it will be necessary to excavate a trench along the cross-section *PCH* in order to expose the rock, so as to set the stakes P_1 and H_1. Then *P* and *H* can be located as usual.

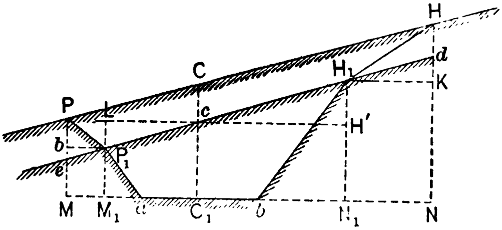


FIG. 151.

It is easily shown, however, that when the layer of earth is uniform,

$$CH' + H_1K = \frac{S'(\frac{1}{2}b + cS)}{S' - S} + \frac{dS'S_1}{S' - S_1}, \quad . \quad . \quad (11)$$

and $$CL + P_1O = \frac{S'(\frac{1}{2}b + cS)}{S' + S} + \frac{dS'S_1}{S' + S_1}. \quad . \quad . \quad (12)$$

In these equations S = side slope of the rock, S_1 = that of the earth, and S' = the slope of the ground; also b = width of the roadbed, c = center depth of rock, and d = that of the earth.

BORROW-PITS.—When the cuts are insufficient to make the fills, or are too far away, material taken from borrow-pits is used. These should be carefully staked out by the engineer so that their contents can be calculated when completed.

It is usual to lay out the area to be used in squares or rectangles, extending one or both sets of lines, so that when the excavation is completed the lines can be readily reproduced in the bottom of the borrow-pit, and heights taken there under the original heights taken upon the surface, thus giving the depths of the excavation.

If the sides of the rectangles in *one* direction are 10 feet or some multiple of 10 feet, and in the other direction 27 feet or some multiple of 27 feet, the computation is very much facilitated.

Borrow-pits should be regularly excavated so that they can be easily computed and will not present an unsightly appearance when abandoned.

Material, when suitable, may be obtained by widening the cuts, provided the fills are accessible and not too far away.

Any surplus material in the cuts should be used in widening the adjacent fills if possible, otherwise it should be deposited where the engineer directs, and is said to be wasted. It should in no case be deposited on the upper side of a cut, unless well removed from the edge, for otherwise it would greatly interfere with the surface ditches, which should always extend along the upper side of a cut, a few feet from the edge, to prevent the surface water from pouring into them.

SHRINKAGE.—In estimating the volumes of cuts and fills to be made from them we have to bear in mind that earths are more compact in fills than in cuts, but rock is less so. The shrinkage in the bank is about as follows for different materials:

Gravel and gravelly earth............	8 per cent.
Gravel and sand.....................	9 “ “
Clay and clayey earth...............	10 “ “
Loam and light sandy soils...........	12 “ “
Loose vegetable soils.................	15 “ “
Puddled clay.........................	25 “ “

Much, however, depends upon the condition of the material when handled, and the mechanical appliances used.

Embankments made with wheelbarrows are very loose and will shrink 15 per cent or more if of a clayey or gravelly nature, and perhaps 20 or even 25 per cent if loamy or of loose vegetable soil.

Embankments formed with wagons do not shrink so much, and those put up by means of drag-scrapers or wheel-scrapers shrink least of all.

In view of shrinkage engineers should have embankments built higher than they are desired to be after settling, and should have the stakes set accordingly.

Thus if it is desired to have an embankment 10 feet high after settling, it should be cross-sectioned and built 11 feet high, say.

Some engineers have the stakes marked 10 feet, say, and direct the contractor to put in an extra foot of earth, but since the volumes of embankments increase more rapidly than their heights this is not advisable.

Retracing the Line.—As the grading is nearing completion points are established on the roadbed from the reference-points, and the line is retraced, setting substantial stakes of convenient heights 100 or 200 feet apart on tangents, and about 50 feet, more or less, on curves.

The chain used should correspond in length with the one used in measuring the line, otherwise the result would be as shown in Chapter VI.

The earthwork must be checked as to widths, depths, etc., and any additional work to be done should be plainly indicated on stakes set for the purpose.

The engineer must see that the grade is properly “dressed,” the side ditches fully opened, so as to secure “continuous” drainage, the surface ditches dug and unobstructed, etc.

Tile drains should be laid in the bottom of the ditches in all cuts except those especially dry. These tiles have a decided effect in keeping the earth on both sides of the ditches dry, thus protecting the roadbed, preventing the earth from the face of the cut from sliding into the ditches, and decreasing the expense of maintenance.

TRACK-LAYING.—Before the track is laid the center line is retraced and short stakes are set, each of which is centered. These stakes may be spaced at 200 feet on tangents, 50 feet on ordinary curves, and 25 feet on very sharp curves.

The ties are then spaced, so many per rail length, leaving, however, the transit points uncovered.

The ties are aligned on one side of the road, and it they are of uniform lengths both ends will be aligned. The rails are then spiked to gauge, the first spikes being driven near a center stake, the center mark of the gauge-bar being kept over the center on the stake.

Upon curves the rails, as a rule, must be bent to conform to the curve before being laid, but this is not necessary for very flat curves, and, moreover, a *track* of light rails, say 50 pounds per yard or less, can be easily sprung into any desired curve.

All the ties to be used should be laid before the rails are placed upon them, otherwise the rails are likely to be bent before the track is surfaced.

Owing to expansion of the rails by heat a space must be left at the rail-joints. The highest temperature of a rail in the sun is about 130° Fahr.

The expansion of iron or steel per 100° is .0007 per foot. Therefore when 30-foot rails are laid at a temperature near the freezing-point (32°), or 100° below the maximum, the space allowed must be at least $.0007 \times 30 = .021$ foot, or .252 inch, or fully one quarter of an inch.

At 80° Fahr., or 50° below the maximum, it need be only half as much.

The space required is, of course, proportional to the length of the rail.

The engineer should provide, or see that the contractor provides, wedges suitable to different temperatures in which the rails are laid, and also see that they are used, for too small a space would result in the rails being forced up by expansion, l a space too large would result in a rough track.

Where sidings are required the necessary frogs and switch-ties should be provided in advance, so that they may be put in place as the main track is laid.

Heavy plank for road-crossings should be laid as soon as the rails are spiked so that the highway travel may not be interrupted.

CULVERTS.—For small openings piping answers an excellent purpose. The ground must be brought to grade and well tamped so as to form a firm and *homogeneous* bed for the pipe. If the natural soil is unsuitable to form a firm bed, a quantity of clay should be provided for it and tamped as above. The *upper* surface of the pipe should be at least 2 feet below grade.

It is generally bad practice to use stone in any way about these culverts, since it destroys the *homogeneity* of the structure and thus invites scouring and destruction of the culvert.

The author has constructed many pipe culverts without the use of a stone, and without, so far as known to him, the loss of a single culvert.

Open wooden culverts are well adapted for somewhat larger openings in low embankments. Any defects in them are easily seen, and they are easily renewed or replaced if desired. Covered culverts are sometimes used in higher embankments, the covers being square timbers 12 or more inches in thickness.

The walls of the culvert should extend 1½ or 2 feet below the surface of the ground, and should be well tied into the bank by timbers embedded for that purpose.

The bed of the culvert should be left *undisturbed*, the openings should be as nearly of a uniform cross-section as possible, having a moderate uniform slope, and their approaches should be as straight and uniform in slope as practicable.

Where these essentials are fully observed and the culverts are properly constructed there is no danger of their washing out. Culverts are often destroyed in consequence of being so small as to partly block up the stream, and thus disturb its easy and natural flow, causing whirls, eddies, washing of banks, etc., and finally undermining and perhaps washing out the culvert.

It is not enough that a culvert is able to discharge all the water that approaches it; for safety it must do so without too much disturbing the natural and normal flow of the water.

Culverts of less than 8 or 10 feet opening should not be constructed in any case.

For the foundation of large culverts piles can often be used to advantage. Such a foundation is comparatively inexpensive; it will last indefinitely and cannot be washed out. Moreover, the structure itself can be so fastened together and to the piles as to be immovable.

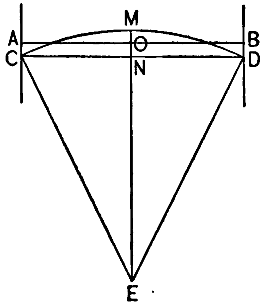
Where good stone is plentiful, arched masonry culverts, though costly, are in some respects the best of all, since if properly built they are very durable and therefore not likely to fail and cause disaster.

The location of a culvert depends somewhat upon the conformation of the ground, it being necessary sometimes (but should be avoided when practicable) to set the culvert on a "skew," in order to avoid too much excavation or to fit, to the best advantage, the thread of the stream.

A stake is set at each corner of the area to be occupied by the culvert, and the cut to be made is marked upon it. The general principles of locating points and lines under various conditions having been fully explained in preceding chapters, including Chapter IV, it is only necessary to apply them to any and every case that may arise.

The location of bridge piers on a curve requires, however, something of a compromise; and what *that* is, and a handy way of doing it, will therefore be explained in this connection.

Let *CD*, Fig. 152, represent the curve between two adjacent piers. Draw the chord *CD*.



Fgi. 152.

The middle ordinate

$$MN = CE \text{ vers } CEM.$$

Take $CA = DB = \frac{1}{2}MN$. *A* and *B* are the centers of the piers.

We observe that *AB* is on a line half-way between the chord *CD* and tangent to the curve at *M*.

Proceed in the same way for other piers.

TUNNELS.—Tunnels, when possible, should be on a tangent throughout, so as to be easily laid out, and to freely admit the light.

The location of Tunnels, like that of Bridges and other structures should of course be such as to render the total cost a minimum for a given service rendered.

The material to be encountered may in some cases be determined with tolerable accuracy by a study of the geology of the adjacent region, but for more accurate information it is necessary to resort to borings.

The alignment of a tunnel is more or less elaborate according to its length and surroundings, and great care should be bestowed upon it. Indeed in all cases where errors would be more or less costly every effort should be made to avoid them by using the best tapes and rods and instruments, in perfect adustment, and by repeating, several times, all observations with transit, level, etc., and all computations. It is of the greatest importance to have the transit revolve in a vertical plane, and to secure this a sensitive bubble-tube should be attached to the horizontal axis of the telescope.

It will usually be necessary to find the distance through the tunnel by triangulation.

By triangulation also, and by direct surveying, the highest point and other high points on the line of the tunnel are found, as well as other points on the prolongation of the line in both directions. These points determine the line, and stations are established at them by means of which any desired point may be located and the work easily laid out.

As the excavation of the tunnel proceeds successive points on the center line are determined, usually on the roof, from which plumb-lines are suspended, thus plainly and constantly defining the line.

Whether tunnels are on tangents or curves the surveying operations connected with them are simple and have been fully explained.

As already stated, great care, repetitions, and numerous and varied checks on the work are important.

For detailed and full information regarding tunnels the reader is referred to Sims' and Drinker's books on the subject, and to the current engineering journals.

CHAPTER XIII.

EXPLANATION OF TABLES AND MISCELLANEOUS TOPICS.

Table I. The radius of a 1° curve $= \dfrac{180 \times 100}{\pi}$.

The radius of a 1′ curve $= \dfrac{180 \times 100 \times 60}{\pi} = 343774.677078.$

To find the radius of any curve divide the radius of a 1′ curve by the number of minutes in the degree of the given curve.

Table II contains tangent offsets for all arcs up to 25 feet, 50 feet, or 100 feet for curves of different degrees.

Tables III, III*a*, and III*b* contain tangential offsets, middle ordinates, and chords for arcs of 100 feet, differing in degree by 10′.

Tables IV and V contain the value of long chords and middle ordinates to long chords for curves from 0° to 20°, differing by 10. These tables are very useful for passing obstacles, for finding the middle point of any arc, for laying out curves, etc., etc.

Table VI contains the various elements of frogs and turnout curves.

Table VII contains the true values of the tangents and externals of a 1′ curve. To find these functions for any other curve divide the tabular numbers corresponding to the given intersection angle by the number of minutes in the degree of the curve.

Thus for a 2° 14′ curve and $V = 28°$ we have

$$\text{tangent} = 85712.7 \div 134 = 639.7,$$

and
$$\text{external} = 10524.2 \div 134 = 78.5.$$

Thus the true result is given by a single operation, whereas the usual tables require *three* operations to find an *approximate* result.

Tables VIII, IX, and X are supposed to be in the most convenient form for use.

Table XI contains the correction *e*, in feet, corresponding to any distance, *D*, due to curvature and refraction.

Tables XII, XIII, and XIV need no explanation.

Tables XV and XVI contain, respectively, the offsets and tangent distances of transition curves, in terms of the degree of the offset curve, and the length of the offset or transition curve.

Thus for an offset curve of 6° and transition curve 200 feet long the offset by Table XV is 1.74. Also, by Table XVI,

$$\frac{S}{2} - AO = .03, \quad \text{or} \quad AO = 100 - .03 = 99.97.$$

Table XVII contains deflection angles for transition curves, for one to five chords, in terms of the angle for the first chord. Thus if the deflection angle for the first chord is $2'.3$, then the angles for the second, third, fourth, and fifth chords are $9'.2$, $20'.7$, $36'.8$, and $57'.5$, respectively.

Table XVIII contains the volumes of earth, etc., for different slopes and bases.

Table XIX applies to all bases having slopes of 1 to 1 or $1\frac{1}{2}$ to 1.

Example.—Let base *ab* (Fig. 135) = 20, slopes 1 to 1.

Let $CD = c = 6.4$. Then $DE = 10$, and $CE = 6.4 + 10 = 16.4$.

$$\begin{array}{lr} \text{Now for 16.4 the table gives, volume for } HKE = & 996 \\ \text{and for 10 the table gives, volume for } abE = & 370 \\ \hline \text{Therefore the desired volume for } abHK = & 626 \end{array}$$

The subgrade volume *abE* being constant for any given base and slope can be taken from the table once for all.

A table designed specially for *any* base can be used as above for any other base having the same slopes.

The remaining tables perhaps require no explanation.

MISCELLANEOUS.

To Gauge a Stream Approximately.—Take some body, a partly filled bottle, for example, that will float nearly submerged, and allow it to do so down a uniform and open stretch of the stream 100 feet in length, and note the time *T* in seconds.

Measure the cross-section A in square feet, then the cubic feet of water that passes per minute is

$$Q = \frac{5000A}{T}.$$

This assumes that the average velocity is $\frac{5}{6}$ of the observed velocity. (See Bowser's Hydromechanics, p. 217.)

Horse-power of falling water = $.00189QH$, in which Q = cubic feet per minute falling H feet.

Transverse Strength of Rectangular Beams.—Let L = length in feet, b = the breadth, and d = depth in inches; w = load at center in pounds, and R = modulus of rupture. Then

$$w = \frac{Rbd^2}{18L}.$$

R is taken from 1000 to 15,000 pounds for wood, and from 10,000 to 15,000 pounds for wrought iron.

If the load is distributed, it will carry twice the above amount. This is useful during construction in deciding whether or not working trains can safely pass over unfinished bridges, culverts, etc.

Safe Bearing-power of Piles.—Let w = weight in pounds of the hammer used in driving a pile, h = the fall in feet, S = the penetration in inches, and R = the bearing-power sought. Then, assuming a factor of safety of 6,

$$R = \frac{2wh}{S + 1},$$

This is the formula of the late Mr. A. M. Wellington, formerly editor of the *Engineering News*. It has the merit of being as trustworthy as any, combined with great simplicity.

TABLES.

Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*	Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*
0′	No rad.	100.00			1° 0′	5729.58		.873	.218
1	343775.		.015	.004	1	5635.65		.887	.222
2	171887.		.029	.007	2	5544.75		.902	.225
3	114592.		.044	.011	3	5456.74		.916	.229
4	85943.7		.058	.015	4	5371.48		.931	.233
5	68754.9		.073	.018	5	5288.84		.945	.236
6	57295.8		.087	.022	6	5208.71	99.998	.960	.240
7	49110.7		.102	.025	7	5130.97		.974	.244
8	42971.8		.116	.029	8	5055.51		.989	.247
9	38197.2		.131	.033	9	4982.24		1.004	.251
10	34377.5		.145	.036	10	4911.07		1.018	.255
11	31252.2		.160	.040	11	4841.90		1.033	.258
12	28647.9		.175	.044	12	4774.65		1.047	.262
13	26444.2		.189	.047	13	4709.24		1.062	.265
14	24555.3		.204	.051	14	4645.60		1.076	.269
15	22918.3		.218	.055	15	4583.66		1.091	.273
16	21485.9		.233	.058	16	4523.35		1.105	.276
17	20222.0		.247	.062	17	4464.61		1.120	.280
18	19098.6		.262	.065	18	4407.87		1.134	.284
19	18093.4		.276	.069	19	4351.58		1.149	.287
20	17188.7		.291	.073	20	4297.18		1.164	.291
21	16370.2		.305	.076	21	4244.13		1.178	.295
22	15626.1		.320	.080	22	4192.37		1.193	.298
23	14946.7		.335	.084	23	4141.86		1.207	.302
24	14323.9		.349	.087	24	4092.56		1.222	.305
25	13751.0		.364	.091	25	4044.41	99.997	1.236	.309
26	13222.1		.378	.095	26	3997.38		1.251	.313
27	12732.4		.393	.098	27	3951.43		1.265	.316
28	12277.7		.407	.102	28	3906.53		1.280	.320
29	11854.3		.422	.105	29	8862.64		1.294	.324
30	11459.2		.436	.109	30	3819.72		1.309	.327
31	11089.5		.451	.113	31	3777.74		1.323	.331
32	10743.0		.465	.116	32	3736.68		1.338	.335
33	10417.4		.480	.120	33	3696.50		1.353	.338
34	10111.0		.495	.124	34	3657.18		1.367	.342
35	9822.13		.509	.127	35	3618.68		1.382	.345
36	9549.30		.524	.131	36	3580.99		1.396	.349
37	9291.21		.538	.135	37	3544.07		1.411	.353
38	9046 71	99.999	.553	.138	38	3507.90		1.425	.356
39	8814.74		.067	.142	39	3472.47		1.440	.360
40	8594.37		.582	.145	40	3437.75	99.996	1.454	.364
41	8384.75		.596	.149	41	3403.71		1.469	.367
42	8185.11		.611	.153	42	3370.34		1.483	.871
43	7994.76		.625	.156	43	3337.62		1.498	.375
44	7813.06		.640	.160	44	3305.53		1.513	.378
45	7639.44		.654	.164	45	3274.04		1.527	.382
46	7473.36		.669	.167	46	3243.16		1.542	.385
47	7314.35		.684	.171	47	3212.85		1.556	.389
48	7161.97		.698	.175	48	3183.10		1.571	.393
49	7015.81		.713	.178	49	3153.90		1.585	.396
50	6875.49		.727	.182	50	3125.22		1.600	.400
51	6740.68		.742	.185	51	3097.07		1.614	.404
52	6611.05		.756	.189	52	3069.42		1.629	.407
53	6486.31		.771	.193	53	3042.25	99.995	1.643	.411
54	6366.20		.785	.196	54	3015.57		1.658	.415
55	6250.45		.800	.200	55	2989.35		1.672	.418
56	6138.83		.814	.204	56	2963.57		1.687	.422
57	6031.13		.829	.207	57	2938.25		1.702	.425
58	5927.15		.844	.211	58	2913.34		1 716	.429
59	5826.69		.858	.215	59	2888.86		1.731	.433
60	5729.58		.673	.218	60	2864.79		1.745	.436

Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*	Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang Off. *t.*	Mid. Ord. *m.*
2° 00′	2864.79		1.745	.436	3° 00′	1909.86		2.617	.654
1	2841.11		1.760	.440	1	1899.31	99.988	2.632	.658
2	2817.83		1.774	.444	2	1888.87		2.646	.662
3	2794.92		1.789	.447	3	1878.55		2.661	.665
4	2772 38		1.803	.451	4	1868.34		2.676	.669
5	2750.20	99.994	1.818	.454	5	1858.24		2.690	.673
6	2728.37		1.832	.457	6	1848.25		2.705	.676
7	2706.89		1.847	.462	7	1838.37		2.719	.680
8	2685.74		1.861	.465	8	1828.59		2.734	.684
9	2664.92		1.876	.468	9	1818.91	99.987	2.748	.687
10	2644.42		1.891	.473	10	1809.34		2.763	.691
11	2624.23		1.905	.476	11	1799.87		2.777	.694
12	2604.35		1.920	.480	12	1790.49		2.792	.698
13	2584.77		1.934	.484	13	1781.22		2.806	.702
14	2565.48		1.949	.487	14	1772.03		2.821	.705
15	2546.48		1.963	.491	15	1762.95		2.835	.709
16	2527.75	99.993	1.978	.494	16	1753.95	99.986	2.850	.713
17	2509.30		1.992	.498	17	1745.05		2.864	.716
18	2491.12		2.007	.502	18	1736.24		2.879	.720
19	2473.20		2.021	.505	19	1727.51		2.894	.724
20	2455.53		2.036	.509	20	1718.87		2.908	.727
21	2438.12		2.050	.513	21	1710.32		2.923	.731
22	2420.95		2.065	.516	22	1701.85		2.937	.734
23	2404.02		2.080	.520	23	1693.47	99.985	2.952	.738
24	2387.32		2.094	.524	24	1685.17		2.966	.742
25	2370.86		2.109	.527	25	1676.95		2.981	.745
26	2354.62	99.992	2.123	.531	26	1668 81		2 995	.749
27	2338.60		2.138	.534	27	1660.75		3.010	.753
28	2322.80		2.152	.538	28	1652.76		3.024	.756
29	2307.21		2.167	.542	29	1644.85		3.039	.760
30	2291.88		2.181	.545	30	1637.02	99.984	3.053	.764
31	2276.65		2.196	.549	31	1629.26		3.068	.767
32	2261.68		2.210	.553	32	1621.58		3.082	.771
33	2246.89		2.225	.556	33	1613.97		3.097	.774
34	2232.30		2 239	.560	34	1606.42		3.112	.778
35	2217.90		2.254	.564	35	1598.95		3.126	.782
36	2203.68	99.991	2.269	.567	36	1591.55		3.141	.785
37	2189.65		2.283	.571	37	1584.22	99.983	3.155	.789
38	2175.79		2.298	.574	38	1576.95		3.170	.793
39	2162.10		2.312	.578	39	1569.75		3.184	.796
40	2148.59		2.327	.582	40	1562.61		3.199	.800
41	2135.25		2.341	.585	41	1555.54		3.213	.804
42	2122.07		2.356	.589	42	1548.53		3.228	.807
43	2109.05		2.370	.593	43	1541.59	99.982	3.242	.811
44	2096.19		2.385	.596	44	1534.71		3.257	.814
45	2083.48	99.990	2.399	.600	45	1527.89		3.271	.818
46	2070.93		2.414	.604	46	1521.13		3.286	.822
47	2058.53		2.428	.607	47	1514.43		3.300	.825
48	2046.28		2.443	.611	48	1507.78		3.315	.829
49	2034.17		2.458	.614	49	1501.20		3.329	.833
50	2022.20		2.472	.618	50	1494.67	99.981	3.344	.836
51	2010.38		2.487	.622	51	1488.20		3.358	.840
52	1998.69		2.501	.625	52	1481.79		3.373	.843
53	1987.14	99.989	2.516	.629	53	1475.43		3.388	.847
54	1975.72		2.530	.633	54	1469.12		3.402	.851
55	1964.43		2.545	.636	55	1462.87		3.417	.854
56	1953.27		2.559	.640	56	1456.67	99.980	3.431	.858
57	1942.23		2.574	.644	57	1450.53		3.446	.862
58	1931.32		2.588	.647	58	1444.43		3.460	.865
59	1920.53		2.603	.651	59	1438.39		3.475	.869
60	1909.86		2.617	.654	60	1432.39		3.489	.87

Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*	Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*
4° 0′	1432.39		3.489	.873	5° 0′	1145.92		4.361	1.091
1	1426.45		3.504	.876	1	1142.11		4.375	1.094
2	1420.56	99.979	3.518	.880	2	1138.33		4.390	1.098
3	1414.71		3.533	.883	3	1134.57		4.404	1.102
4	1408.91		3.547	.887	4	1130.84	99.967	4.419	1.105
5	1403.16		3.562	.891	5	1127.18		4.433	1.109
6	1397.46		3.576	.894	6	1123.45		4.448	1.112
7	1391.80	99.978	3.591	.898	7	1119.79		4.462	1.116
8	1386.19		3.605	.902	8	1116.15		4.477	1.120
9	1380.62		3.620	.905	9	1112.54	99.966	4.491	1.123
10	1375.10		3.635	.909	10	1108.95		4.506	1.127
11	1369.62		3.649	.913	11	1105.38		4.520	1.131
12	1364.19		3.664	.916	12	1101.84		4.535	1.134
13	1358.79	99.977	3.678	.920	13	1098.32	99.965	4.549	1.138
14	1353.44		3.693	.923	14	1094.82		4.564	1.142
15	1348.14		3.707	.927	15	1091.35		4.578	1.145
16	1342.87		3.722	.931	16	1087.89		4.593	1.149
17	1337.64		3.736	.934	17	1084.46		4.607	1.152
18	1332.46		3.751	.938	18	1081.05	99.964	4.622	1.156
19	1327.32	99.976	3.765	.942	19	1077.66		4.636	1.160
20	1322.21		3.780	.945	20	1074.30		4.651	1.163
21	1317.14		3.794	.949	21	1070.95		4.665	1.167
22	1312.12		3.809	.953	22	1067.62	99.963	4.680	1.171
23	1307.13		3.823	.956	23	1064.32		4.694	1.174
24	1302.18	99.975	3.838	.960	24	1061.03		4.709	1.178
25	1297.26		3.852	.963	25	1057.77		4.723	1.182
26	1292.39		3.867	.967	26	1054.52		4.738	1.185
27	1287.55		3.881	.971	27	1051.30	99.962	4.752	1.189
28	1282.74		3.896	.974	28	1048.09		4.767	1.192
29	1277.97	99.974	3.910	.978	29	1044.91		4.781	1.196
30	1273.24		3.925	.982	30	1041.74		4.796	1.200
31	1268.54		3.939	.985	31	1038.59	99.961	4.810	1.203
32	1263.88		3.954	.989	32	1035.47		4.825	1.207
33	1259.25		3.969	.993	33	1032.36		4.840	1.211
34	1254.65		3.983	.996	34	1029.27		4.854	1.214
35	1250.09	99.973	3.998	1.000	35	1026.19	99.960	4.869	1.218
36	1245.56		4.012	1.003	36	1023.14		4.883	1.221
37	1241.06		4.027	1.007	37	1020.10		4.898	1.225
38	1236.60		4.041	1.011	38	1017.08		4.912	1.229
39	1232.17		4.056	1.014	39	1014.08	99.959	4.927	1.232
40	1227.77	99.972	4.070	1.018	40	1011.10		4.941	1.236
41	1223.40		4.085	1.022	41	1008.14		4.956	1.240
42	1219.06		4.099	1.025	42	1005.19		4.970	1.243
43	1214.75		4.114	1.029	43	1002.26		4.985	1.247
44	1210.47		4.128	1.033	44	999.345	99.958	4.999	1.251
45	1206.23	99.971	4.143	1.036	45	996.448		5.014	1.254
46	1202.01		4.157	1.040	46	993.569		5.028	1.258
47	1197.82		4.172	1.043	47	990.705		5.043	1.261
48	1193.66		4.186	1.047	48	987.858	99.957	5.057	1.265
49	1189.53		4.201	1.051	49	985.028		5.072	1.269
50	1185.43	99.970	4.215	1.054	50	982.218		5.086	1.272
51	1181.36		4.230	1.058	51	979.415		5.101	1.276
52	1177.31		4.244	1.062	52	976.633	99.956	5.115	1.280
53	1173.29		4.259	1.065	53	973.866		5.130	1.283
54	1169.30		4.273	1.069	54	971.115		5.144	1.287
55	1165.34	99.969	4.288	1.072	55	968.379		5.159	1.291
56	1161.40		4.302	1.076	56	965.659	99.955	5.173	1.294
57	1157.49		4.317	1.080	57	962.954		5.188	1.298
58	1153.61		4.332	1.083	58	960.264		5.202	1.301
59	1149.75	99.968	4.346	1.087	59	957.590		5.217	1.305
60	1145.92		4.361	1.091	60	954.930	99.954	5.231	1.309

Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*	Degree *D.*	Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*
6° 00′	954.930		5.231	1.309	7° 00′	818.511		6.101	1.527
1	952.284		5.246	1.312	1	816.567		6.116	1.530
2	949.654		5.260	1.316	2	814.632	99.937	6.130	1.534
3	947.038		5.275	1.320	3	812.706		6.145	1.538
4	944.436	99.953	5.289	1.323	4	810.789		6.159	1.541
5	941.848		5.304	1.327	5	808.882	99.936	6.174	1.545
6	939.275		5.318	1.330	6	806.983		6.188	1.548
7	936.716		5.333	1.334	7	805.093		6.202	1.552
8	934.170	99.952	5.347	1.338	8	803.212	99.935	6.217	1.556
9	931.639		5.362	1.341	9	801.340		6.231	1.559
10	929.121		5.376	1.345	10	799.476		6.246	1.563
11	926.616	99.951	5.391	1.349	11	797.621		6.260	1.567
12	924.125		5.405	1.352	12	795.775	99.934	6.275	1.570
13	921.648		5.420	1.356	13	793.937		6.289	1.574
14	919.184		5.434	1.360	14	792.108		6.304	1.578
15	916.732	99.950	5.449	1.363	15	790.287	99.933	6.318	1.581
16	914.294		5.463	1.367	16	788.474		6.333	1.585
17	911.869		5.478	1.370	17	786.670		6.347	1.588
18	909.457		5.492	1.374	18	784.874	99.932	6.362	1.592
19	907.057	99.949	5.507	1.378	19	783.058		6.376	1.596
20	904.670		5.521	1.381	20	781.306		6.391	1.599
21	902.296		5.536	1.385	21	779.534	99.931	6.405	1.603
22	899.934		5.550	1.389	22	777.771		6.420	1.607
23	897.584	99.948	5.565	1.392	23	776.015		6.434	1.610
24	895.247		5.579	1.396	24	774.267		6.449	1.614
25	892.921		5.594	1.400	25	772.527	99.930	6.463	1.618
26	890.608	99.947	5.608	1.403	26	770.795		6.478	1.621
27	888.307		5.623	1.407	27	769.071		6.492	1.625
28	886.017		5.637	1.410	28	767.354	99.929	6.507	1.628
29	883.740		5.652	1.414	29	765.645		6.521	1.632
30	881.474	99.946	5.666	1.418	30	763.944		6.536	1.636
31	879.219		5.681	1.421	31	762.250	99.928	6.550	1.639
32	876.976		5.695	1.425	32	760.563		6.565	1.643
33	874.745		5.710	1.429	33	758.885		6.579	1.647
34	872.525	99.945	5.724	1.432	34	757.213	99.927	6.594	1.651
35	870.316		5.739	1.436	35	755.549		6.608	1.654
36	868.118		5.753	1.439	36	753.892		6.623	1.657
37	865.931	99.944	5.768	1.443	37	752.242	99.926	6.637	1.661
38	868.755		5.782	1.447	38	750.600		6.651	1.665
39	861.591		5.797	1.451	39	748.964		6.666	1.668
40	859.437		5.811	1.454	40	747.336	99.925	6.680	1.672
41	857.293	99.943	5.826	1.458	41	745.715		6.695	1.676
42	855.161		5.840	1.461	42	744.101		6.709	1.679
43	853.039		5.855	1.465	43	742.494	99.924	6.724	1.683
44	850.927	99.942	5.869	1.469	44	740.894		6.738	1.687
45	848.826		5.884	1.472	45	739.300		6.753	1.690
46	846.736		5.898	1.476	46	737.714	99.923	6.767	1.694
47	844.655		5.913	1.479	47	736.134		6.782	1.697
48	842.585	99.941	5.927	1.483	48	734.561		6.796	1.701
49	840.525		5.942	1.487	49	732.997	99.922	6.811	1.705
50	838.475		5.956	1.490	50	731.435		6.825	1.708
51	836.435	99.940	5.971	1.494	51	729.883		6.840	1.712
52	834.405		5.985	1.498	52	728.336	99.921	6.854	1.716
53	832.384		6.000	1.501	53	726.796		6.869	1.719
54	830.374		6.014	1.505	54	725.263		6.883	1.723
55	828.373	99.989	6.029	1.509	55	723.736	99.920	6.898	1.726
56	826.381		6.043	1.512	56	722.216		6.912	1.730
57	824.400		6.058	1.516	57	720.702		6.927	1.734
58	822.427	99.938	6.072	1.519	58	719.194	99.919	6.941	1.737
59	820.465		6.087	1.523	59	717.692		6.956	1.741
60	818.511		6.101	1.527	60	716.197		6.970	1.745

Degree *D*.	Radius *R*.	Chord 1 Sta.	Tang. Off. *t*.	Mid. Ord. *m*.	Degree *D*.	Radius *R*.	Chord 1 Sta.	Tang. Off. *t*.	Mid. Ord. *m*.
8° 00′	716.197		6.970	1.745	9° 00′	636.620		7.838	1.963
1	714.708	99.918	6.984	1.748	1	635.443		7.852	1.966
2	713.225		6.999	1.752	2	634.271	99.896	7.867	1.970
3	711.749		7.013	1.756	3	633.103		7.881	1.973
4	710.278	99.917	7.028	1.759	4	631.939		7.896	1.977
5	708.814		7.042	1.763	5	630.779	99.895	7.910	1.981
6	707.355		7.057	1.766	6	629.624		7.925	1.984
7	705.903	99.916	7.071	1.770	7	628.473		7.939	1.988
8	704.456		7.086	1.774	8	627.326	99.894	7.953	1.992
9	703.016		7.100	1.777	9	626.183		7.968	1.995
10	701.581	99.915	7.115	1.781	10	625.045	99.893	7.982	1.999
11	700.152		7.129	1.785	11	623.910		7.997	2.002
12	698.729		7.144	1.788	12	622.780		8.011	2.006
13	697.312	99.914	7.158	1.792	13	621.654	99.892	8.026	2.010
14	695.900		7.173	1.795	14	620.532		8.040	2.013
15	694.494		7.187	1.799	15	619.414	99.891	8.055	2.017
16	693.094	99.913	7.202	1.803	16	618.300		8.069	2.021
17	691.700		7.216	1.806	17	617.190		8.084	2.024
18	690.311		7.230	1.810	18	616.084	99.890	8.098	2.028
19	688.927	99.912	7.245	1.814	19	614.982		8.112	2.031
20	687.549		7.259	1.817	20	613.883	99.889	8.127	2.035
21	686.177		7.274	1.820	21	612.789		8.141	2.039
22	684.810	99.911	7.288	1.825	22	611.699		8.156	2.042
23	683.449		7.303	1.828	23	610.612	99.888	8.170	2.046
24	682.093	99.910	7.317	1.832	24	609.530		8.185	2.050
25	680.742		7.332	1.835	25	608.451	99.887	8.199	2.053
26	679.397		7.346	1.839	26	607.376		8.214	2.057
27	678.057	99.909	7.361	1.843	27	606.305		8.228	2.061
28	676.722		7.375	1.846	28	605.237	99.886	8.242	2.064
29	675.392		7.390	1.850	29	604.173		8.257	2.068
30	674.068	99.908	7.404	1.854	30	603.113	99.885	8.271	2.071
31	672.749		7.419	1.857	31	602.057		8.286	2.075
32	671.435		7.433	1.861	32	601.005		8.300	2.079
33	670.126	99.907	7.447	1.864	33	599.956	99.884	8.315	2.082
34	668.822		7.462	1.868	34	598.911		8.329	2.086
35	667.524		7.476	1.872	35	597.869	99.883	8.344	2.090
36	666.230	99.906	7.491	1.875	36	596.831		8.358	2.093
37	664.941		7.505	1.879	37	595.797		8.372	2.097
38	663.658	99.905	7.520	1.883	38	594.766	99.882	8.387	2.100
39	662.379		7.534	1.886	39	593.739		8.401	2.104
40	661.105		7.549	1.890	40	592.715		8.416	2.108
41	659.836	99.904	7.563	1.894	41	591.695	99.881	8.430	2.111
42	658.572		7.578	1.897	42	590.678		8.445	2.115
43	657.313		7.592	1.901	43	589.665	99.880	8.459	2.119
44	656.059	99.903	7.607	1.904	44	588.655		8.474	2.122
45	654.809		7.621	1.908	45	587.649	99.879	8.488	2.126
46	653.564	99.902	7.635	1.912	46	586.646		8.502	2.129
47	652.324		7.650	1.915	47	585.647		8.517	2.133
48	651.089		7.664	1.919	48	584.651	99.878	8.531	2.137
49	649.858	99.901	7.679	1.923	49	583.658		8.546	2.140
50	648.631		7.693	1.926	50	582.669	99.877	8.560	2.144
51	647.410		7.708	1.930	51	581.683		8.575	2.148
52	646.193	99.900	7.722	1.933	52	580.700		8.589	2.151
53	644.980		7.737	1.937	53	579.721	99.876	8.603	2.155
54	643.773	99.899	7.751	1.941	54	578.745		8.618	2.159
55	642.569		7.766	1.944	55	577.773	99.875	8.632	2.162
56	641.371		7.780	1.948	56	576.803		8.647	2.166
57	640.176	99.898	7.794	1.952	57	575.837	99.874	8.661	2.169
58	638.986		7.809	1.955	58	574.874		8.676	2.173
59	637.801		7.823	1.959	59	573.911		8.690	2.177
60	636.620	99.897	7.838	1.963	60	572.958	99.873	8.705	2.180

Degree *D.*		Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*	Degree *D.*		Radius *R.*	Chord 1 Sta.	Tang. Off. *t.*	Mid. Ord. *m.*
10°	00′	572.958	99.873	8.705	2.180	20°	00′	286.479	99.498	17.277	4.352
	10	568.565	99.869	8.849	2.217		10	284.111	99.485	17.418	4.388
	20	554.475	99.864	8.993	2.253		20	281.783	99.476	17.559	4.424
	30	545.674	99.860	9.137	2.289		30	279.492	99.467	17.700	4.460
	40	537.148	99.856	9.282	2.325		40	277.238	99.459	17.840	4.497
	50	528.884	99.851	9.426	2.362		50	275.020	99.450	17.981	4.533
11	00	520.871	99.847	9.570	2.398	21	00	272.837	99.441	18.122	4.569
	10	513.097	99.842	9.714	2.434		10	270.689	99.432	18.262	4.605
	20	505.551	99.837	9.858	2.471		20	268.574	99.423	18.403	4.641
	30	498.224	99.832	10.002	2.507		30	266.492	99.414	18.543	4.677
	40	491.107	99.827	10.146	2.543		40	264.442	99.405	18.683	4.713
	50	484.190	99.822	10.290	2.579		50	262.423	99.396	18.824	4.749
12	00	477.465	99.817	10.434	2.616	22	00	260.435	99.387	18.964	4.785
	10	470.924	99.812	10.578	2.652		10	258.477	99.378	19.104	4.821
	20	464.560	99.807	10.721	2.688		20	256.548	99.368	19.244	4.857
	30	458.366	99.802	10.865	2.724		30	254.648	99.359	19.384	4.893
	40	452.335	99.797	11.009	2.761		40	252.775	99.349	19.524	4.929
	50	446.461	99.791	11.152	2.797		50	250.930	99.340	19.664	4.965
13	00	440.737	99.786	11.296	2.833	23	00	249.112	99.330	19.803	5.001
	10	435.158	99.780	11.440	2.869		10	247.320	99.320	19.943	5.037
	20	429.718	99.775	11.583	2.906		20	245.553	99.310	20.082	5.073
	30	424.413	99.769	11.727	2.942		30	243.812	99.301	20.222	5.109
	40	419.237	99.763	11.870	2.978		40	242.095	99.291	20.361	5.145
	50	414.186	99.757	12.013	3.014		50	240.402	99.281	20.500	5.181
14	00	409.256	99.751	12.157	3.051	24	00	238.732	99.271	20.639	5.217
	10	404.441	99.745	12.300	3.087		10	237.086	99.260	20.779	5.253
	20	399.738	99.739	12.443	3.123		20	235.462	99.250	20.918	5.289
	30	395.143	99.733	12.586	3.159		30	233.860	99.240	21.056	5.325
	40	390.653	99.727	12.729	3.195		40	232.280	99.230	21.195	5.361
	50	386.264	99.721	12.872	3.232		50	230.721	99.219	21.334	5.397
15	00	381.972	99.715	13.015	3.268	25	00	229.183	99.208	21.473	5.433
	10	377.774	99.708	13.158	3.304		10	227.665	99.198	21.611	5.468
	20	373.668	99.702	13.301	3.340		20	226.168	99.187	21.750	5.504
	30	369.650	99.695	13.444	3.376		30	224.689	99.177	21.888	5.540
	40	365.718	99.689	13.587	3.413		40	223.230	99.166	22.026	5.576
	50	361.868	99.682	13.729	3.449		50	221.790	99.155	22.164	5.612
16	00	358.099	99.675	13.872	3.485	26	00	220.368	99.144	22.303	5.648
	10	354.407	99.669	14.015	3.521		10	218.965	99.133	22.441	5.684
	20	350.790	99.662	14.157	3.557		20	217.579	99.122	22.578	5.720
	30	347.247	99.655	14.300	3.594		30	216.210	99.111	22.716	5.756
	40	343.775	99.648	14.442	3.630		40	214.859	99.100	22.854	5.792
	50	340.371	99.641	14.584	3.667		50	213.525	99.089	22.992	5.827
17	00	337.034	99.634	14.727	3.702	27	00	212.207	99.077	23.129	5.863
	10	333.762	99.626	14.869	3.738		10	210.905	99.066	23.267	5.899
	20	330.553	99.619	15.011	3.774		20	209.619	99.054	23.404	5.935
	30	327.404	99.612	15.153	3.810		30	208.348	99.043	23.541	5.971
	40	324.316	99.604	15.295	3.847		40	207.093	99.031	23.678	6.007
	50	321.285	99.597	15.437	3.883		50	205.853	99.020	23.815	6.042
18	00	318.310	99.589	15.579	3.919	28	00	204.628	99.008	23.952	6.078
	10	315.390	99.582	15.721	3.955		10	203.417	98.996	24.089	6.114
	20	312.522	99.574	15.863	3.991		20	202.220	98.984	24.226	6.150
	30	309.707	99.566	16.005	4.027		30	201.038	98.972	24.362	6.186
	40	306.942	99.558	16.146	4.063		40	199.869	98.960	24.499	6.222
	50	304.225	99.550	16.288	4.100		50	198.714	98.948	24.635	6.257
19	00	301.557	99.542	16.429	4.136	29	00	197.572	98.936	24.772	6.293
	10	298.935	99.534	16.571	4.172		10	196.443	98.924	24.908	6.329
	20	296.357	99.526	16.712	4.208		20	195.327	98.911	25.044	6.365
	30	293.825	99.518	16.853	4.244		30	194.223	98.899	25.180	6.400
	40	291.334	99.510	16.995	4.280		40	193.132	98.887	25.316	6.436
	50	288.886	99.501	17.136	4.316		50	192.053	98.874	25.452	6.472
20	00	286.479	99.493	17.277	4.352	30	00	190.986	98.862	25.587	6.508

Arc.	Curve. 1°	2°	3°	4°	5°	Arc.	Curve. 1°	2°	3°	4°	5°
1						51	.227	.454	.681	.908	1.135
2		.001	.001	.001	.002	52	.236	.472	.708	.944	1.180
3	.001	.002	.002	.003	.004	53	.245	.490	.735	.980	1.225
4	.001	.003	.004	.005	.007	54	.254	.509	.763	1.018	1.272
5	.002	.004	.007	.009	.011	55	.264	.528	.792	1.058	1.320
6	.003	.006	.009	.013	.016	56	.274	.547	.821	1.095	1.368
7	.004	.009	.013	.017	.021	57	.284	.567	.851	1.134	1.417
8	.006	.011	.017	.022	.028	58	.294	.587	.881	1.174	1.468
9	.007	.014	.021	.028	.035	59	.304	.608	.911	1.215	1.519
10	.009	.017	.026	.035	.044	60	.314	.628	.942	1.256	1.570
11	.011	.021	.032	.042	.053	61	.325	.649	.974	1.299	1.623
12	.013	.025	.038	.050	.063	62	.335	.671	1.006	1.342	1.677
13	.015	.029	.044	.059	.074	63	.346	.693	1.039	1.385	1.731
14	.017	.034	.051	.068	.086	64	.357	.715	1.072	1.429	1.787
15	.020	.039	.059	.079	.098	65	.369	.737	1.106	1.475	1.843
16	.022	.045	.067	.089	.112	66	.380	.760	1.140	1.520	1.900
17	.025	.050	.076	.101	.126	67	.392	.783	1.175	1.567	1.958
18	.028	.057	.085	.113	.141	68	.404	.807	1.210	1.614	2.017
19	.032	.063	.095	.126	.158	69	.415	.831	1.246	1.662	2.076
20	.035	.070	.105	.140	.175	70	.428	.855	1.283	1.710	2.137
21	.038	.077	.115	.154	.192	71	.440	.880	1.320	1.759	2.199
22	.042	.084	.127	.169	.211	72	.452	.905	1.357	1.809	2.261
23	.046	.092	.138	.185	.231	73	.465	.930	1.395	1.860	2.324
24	.050	.101	.151	.201	.251	74	.478	.956	1.433	1.911	2.388
25	.055	.109	.164	.218	.273	75	.491	.982	1.472	1.963	2.453
26	.059	.118	.177	.236	.295	76	.504	1.008	1.512	2.016	2.519
27	.064	.127	.191	.254	.318	77	.517	1.035	1.552	2.069	2.586
28	.068	.137	.205	.274	.342	78	.531	1.062	1.593	2.123	2.654
29	.073	.147	.220	.294	.367	79	.545	1.089	1.634	2.178	2.722
30	.079	.157	.236	.314	.393	80	.559	1.117	1.675	2.233	2.791
31	.084	.168	.252	.335	.419	81	.573	1.145	1.717	2.290	2.862
32	.089	.179	.268	.357	.447	82	.587	1.174	1.760	2.347	2.933
33	.095	.190	.285	.380	.475	83	.601	1.202	1.803	2.404	3.005
34	.102	.202	.303	.404	.504	84	.616	1.231	1.847	2.462	3.077
35	.107	.214	.321	.428	.534	85	.630	1.261	1.891	2.521	3.151
36	.113	.226	.339	.452	.565	86	.645	1.291	1.936	2 581	3.226
37	.119	.239	.358	.478	.597	87	.661	1.321	1.981	2.641	3.301
38	.126	.252	.378	.504	.630	88	.676	1.352	2.027	2.702	3.377
39	.133	.265	.398	.531	.664	89	.691	1.382	2.073	2.764	3.454
40	.140	.279	.419	.558	.698	90	.707	1.414	2.120	2.827	3.532
41	.147	.293	.440	.587	.733	91	.723	1.445	2.167	2.890	3.611
42	.154	.308	.462	.616	.769	92	.739	1.477	2.215	2.954	3.691
43	.161	.323	.484	.645	.807	93	.755	1.509	2.263	3.018	3.772
44	.169	.338	.507	.675	.845	94	.771	1.542	2.313	3.083	3.853
45	.177	.353	.530	.707	.883	95	.788	1.575	2.362	3.149	3.936
46	.185	.369	.554	.739	.923	96	.804	1.608	2.412	3.216	4.019
47	.193	.386	.578	.771	.964	97	.821	1 642	2.463	3.283	4.103
48	.201	.402	.603	.804	1.005	98	.838	1.676	2.514	3.351	4.188
49	.210	.419	.629	.838	1.047	99	.855	1.710	2.565	3.420	4 274
50	.218	.436	.654	.873	1.090	100	.873	1.745	2.617	3.489	4.361

Arc.	Curve.									
	6°	7°	8°	9°	10°	11°	12°	18°	14°	15°
1	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
2	.002	.002	.002	.003	.003	.004	.004	.005	.005	.005
3	.005	.005	.006	.007	.008	.009	.008	.010	.011	.011
4	.008	.009	.011	.013	.014	.015	.017	.018	.019	.020
5	.013	.015	.017	.020	.022	.024	.026	.028	.031	.032
6	.019	.022	.025	.028	.031	.035	.038	.041	.044	.047
7	.026	.030	.034	.038	.043	.047	.051	.056	.060	.064
8	.034	.039	.045	.050	.056	.061	.067	.073	.078	.084
9	.042	.049	.057	.064	.071	.078	.085	.092	.099	.106
10	.052	.061	.070	.079	.087	.096	.105	.113	.122	.131
11	.063	.074	.084	.095	.106	.116	.127	.137	.148	.158
12	.075	.088	.101	.118	.126	.138	.151	.163	.176	.188
13	.088	.103	.118	.133	.147	.162	.177	.192	.206	.221
14	.103	.120	.137	.154	.171	.188	.205	.222	.239	.257
15	.118	.137	.157	.177	.196	.216	.236	.255	.275	.294
16	.134	.156	.179	.901	.223	.246	.268	.290	.313	.335
17	.151	.177	.202	.227	.252	.277	.303	.328	.353	.378
18	.170	.198	.226	.254	.283	.311	.339	.368	.396	.424
19	.189	.221	.252	.284	.315	.346	.378	.409	.441	.472
20	.909	.244	.279	.314	.349	.384	.419	.454	.489	.523
21	.231	.269	.308	.346	.385	.423	.462	.500	.539	.577
22	.253	.296	.338	.380	.422	.465	.507	.549	.591	.633
23	.277	.323	.369	.415	.462	.508	.554	.600	.646	.692
24	.302	.352	.402	.452	.503	.553	.603	.653	.704	.754
25	.327	.382	.436	.491	.545	.600	.654	.709	.763	.818
26	.354	.413	.472	.531	.590	.649	.708	.767	.826	.885
27	.382	.445	.509	.572	.636	.700	.763	.827	.890	.954
28	.410	.479	.547	.616	.684	.752	.821	.889	.957	1.026
29	.440	.514	.587	.660	.734	.807	.880	.954	1.027	1.100
30	.471	.550	.628	.707	.785	.864	.942	1.021	1.099	1.177
31	.503	.587	.671	.755	.838	.922	1.006	1.090	1.174	1.257
32	.536	.625	.715	.804	.893	.983	1.072	1.161	1 250	1 340
33	.570	.665	.760	.855	.950	1.045	1.140	1.235	1.330	1.424
34	.605	.706	.807	.908	1.009	1.109	1.210	1.311	1.412	1.512
35	.641	.748	.855	.962	1.069	1.174	1.282	1.389	1.496	1.602
36	.679	.792	.904	1.018	1.131	1.244	1.357	1.469	1.582	1.695
37	.717	.836	.955	1.075	1.194	1.314	1.433	1.552	1.671	1.790
38	.756	.882	1.008	1.134	1.260	1.386	1.511	1.637	1.763	1.889
39	.796	.929	1.061	1.194	1.327	1.459	1.592	1.724	1.857	1.989
40	.838	.977	1.117	1.256	1.396	1.535	1.675	1.814	1.953	2.092
41	.880	1.027	1.173	1.320	1.466	1.613	1.759	1.906	2.052	2.198
42	.923	1.077	1.231	1.385	1.539	1.692	1.846	2.000	2.153	2.307
43	.968	1.129	1.290	1.452	1.613	1.774	1.935	2.096	2.257	2.417
44	1.014	1.182	1.351	1.520	1.689	1.857	2.026	2.194	2.363	2.531
45	1.060	1.237	1.418	1.590	1.766	1.943	2.119	2.295	2.472	2.647
46	1.108	1.292	1.477	1.661	1.846	2.030	2.214	2.398	2.582	2.766
47	1.156	1.349	1.541	1.734	1.927	2.119	2.311	2.504	2.696	2.888
48	1.206	1.407	1.608	1.809	2.009	2.210	2.411	2.611	2.812	3.012
49	1.257	1.466	1.675	1.885	2 094	2.303	2.512	2.721	2.930	3.138
50	1.309	1.527	1.745	1.962	2.180	2.398	2.616	2.833	3.051	3.268

Arc.	Curve.									
	16°	17°	18°	19°	20°	21°	22°	23°	24°	25°
1	.001	.001	.002	.002	.002	.002	.002	.002	.002	.002
2	.006	.006	.006	.007	.007	.007	.008	.008	.008	.009
3	.013	.013	.014	.015	.016	.016	.017	.018	.019	.020
4	.022	.024	.025	.027	.028	.029	.031	.032	.034	.035
5	.035	.037	.039	.041	.044	.046	.048	.050	.052	.055
6	.050	.053	.057	.060	.063	.066	.069	.072	.075	.078
7	.068	.073	.077	.081	.086	.090	.094	.098	.103	.107
8	.089	.095	.101	.106	.112	.117	.123	.128	.134	.140
9	.113	.120	.127	.134	.141	.148	.155	.163	.170	.177
10	.140	.148	.157	.166	.175	.183	.192	.201	.209	.218
11	.169	.179	.190	.201	.211	.222	.232	.243	.253	.264
12	.201	.214	.226	.239	.251	.264	.276	.289	.302	.314
13	.236	.251	.265	.280	.294	.310	.324	.339	.354	.369
14	.274	.291	.308	.325	.342	.360	.376	.393	.410	.427
15	.314	.334	.353	.373	.393	.412	.432	.451	.471	.491
16	.357	.379	.402	.424	.447	.469	.491	.514	.536	.558
17	.403	.429	.454	.479	.504	.530	.555	.580	.605	.630
18	.452	.480	.509	.537	.565	.594	.622	.650	.678	.706
19	.504	.535	.567	.598	.630	.661	.693	.724	.756	.787
20	.558	.593	.628	.663	.698	.733	.768	.802	.837	.872
21	.616	.654	.692	.731	.769	.808	.846	.885	.923	.961
22	.676	.718	.760	.802	.844	.886	.929	.971	1.013	1.055
23	.738	.784	.831	.877	.923	.969	1.015	1.061	1.107	1.153
24	.804	.854	.904	.955	1.005	1.055	1.105	1.155	1.205	1.255
25	.872	.927	.981	1.036	1.090	1.145	1.199	1.253	1.307	1.362

TABLE III.—TANGENT OFFSETS ARC 100 FEET.

Degree of Curve.	0′	10′	20′	30′	40′	50′
0°	.000	.145	.291	.436	.582	.727
1	.873	1.018	1.164	1.309	1.454	1.600
2	1.745	1.891	2.036	2.181	2.327	2.472
3	2.617	2.763	2.908	3.053	3.199	3.344
4	3.489	3.635	3.780	3.925	4.070	4.215
5	4.361	4.506	4.651	4.796	4.941	5.086
6	5.231	5.376	5.521	5.666	5.811	5.956
7	6.101	6.246	6.391	6.536	6.680	6.825
8	6.970	7.115	7.259	7.404	7.549	7.693
9	7.838	7.982	8.127	8.271	8.416	8.560
10	8.705	8.849	8.993	9.137	9.282	9.426
11	9.570	9.714	9.858	10.01	10.15	10.29
12	10.43	10.58	10.72	10.86	11.01	11.15
13	11.30	11.44	11.58	11.73	11.87	12.01
14	12.16	12.30	12.44	12.59	12.73	12.87
15	13.01	13.16	13.30	13.44	13.59	13.73
16	13.87	14.01	14.16	14.30	14.44	14.58
17	14.73	14.87	15.01	15.15	15.30	15.44
18	15.58	15.72	15.86	16.00	16.15	16.29
19	16.43	16.57	16.71	16.85	17.00	17.14
20	17.28	17.42	17.56	17.70	17.84	17.98
21	18.12	18.26	18.40	18.54	18.68	18.82
22	18.96	19.10	19.24	19.38	19.52	19.66
23	19.80	19.94	20.08	20.22	20.36	20.50
24	20.64	20.78	20.92	21.06	21.20	21.33
25	21.47	21.61	21.75	21.89	22.03	22.16
26	22.30	22.44	22.58	22.72	22.85	22.99
27	23.13	23.27	23.40	23.54	23.68	23.82
28	23.95	24.09	24.23	24.36	24.50	24.64
29	24.77	24.91	25.04	25.18	25.32	25.45

TABLE IIIA.—MIDDLE ORDINATES ARC 100 FEET.

Degree of Curve.	0′	10′	20′	30′	40′	50′
0°	.000	.036	.073	.109	.145	.182
1	.218	.255	.291	.327	.364	.400
2	.436	.473	.509	.545	.582	.618
3	.654	.691	.727	.764	.800	.836
4	.873	.909	.945	.982	1.018	1.054
5	1.091	1.127	1.163	1.200	1.236	1.272
6	1.309	1.345	1.381	1.418	1.454	1.490
7	1.527	1.563	1.599	1.636	1.672	1.708
8	1.745	1.781	1.817	1.854	1.890	1.926
9	1.963	1.999	2.035	2.071	2.108	2.144
10	2.180	2.217	2.253	2.289	2.325	2.362
11	2.398	2.434	2.471	2.507	2.543	2.579
12	2.616	2.652	2.688	2.724	2.761	2.797
13	2.833	2.869	2.906	2.942	2.978	3.014
14	3.051	3.087	3.123	3.159	3.195	3.232
15	3.268	3.304	3.340	3.376	3.413	3.449
16	3.485	3.521	3.557	3.594	3.630	3.667
17	3.702	3.738	3.774	3.810	3.847	3.883
18	3.919	3.955	3.991	4.027	4.063	4.100
19	4.136	4.172	4.208	4.244	4.280	4.316
20	4.352	4.388	4.424	4.460	4.497	4.533
21	4.569	4.605	4:641	4.677	4.713	4.749
22	4.785	4.821	4.857	4.893	4.929	4.965
23	5.001	5.037	5.073	5.109	5.145	5.181
24	5.217	5.253	5.289	5.325	5.361	5.897
25	5.433	5.468	5.504	5.540	5.576	5.612
26	5.648	5.684	5.720	5.756	5.792	5.827
27	5.863	5.899	5.935	5.971	6.007	6.042
28	6.078	6.114	6.150	6.186	6.222	6.257
29	6.293	6.329	6.365	6.400	6.436	6.472

TABLE IIIB.—CHORDS FOR ARCS 100 FEET.

Degree of Curve.	0′	10′	20′	30′	40′	50′
0°	100.000	100.000	100.000	100.000	99.999	99.999
1	99.999	99.998	99.998	99.997	99.996	99.996
2	99.995	99.994	99.993	99.992	99.991	99.990
3	99.989	99.987	99.986	99 984	99.983	99.981
4	99.980	99.978	99.976	99.974	99.972	99 970
5	99.968	99.966	99.964	99.962	99.959	99.957
6	99.954	99.952	99.949	99.946	99.944	99.941
7	99.938	99.935	99.932	99.929	99.925	99.922
8	99.919	99.915	99.912	99.908	99.905	99.901
9	99.897	99.893	99.889	99.885	99.882	99.877
10	99.873	99.869	99.864	99.860	99.856	99.851
11	99.847	99.842	99.836	99.832	99.827	99.822
12	99.817	99.812	99.807	99.802	99.797	99.791
13	99.786	99.780	99.775	99.769	99.763	99.757
14	99.751	99.745	99.739	99.783	99.727	99.721
15	99.715	99.708	99.702	99.695	99.689	99.682
16	99.675	99.669	99.662	99.655	99.648	99.641
17	99.634	99.626	99.619	99.612	99.604	99.597
18	99.589	99.582	99.574	99 566	99.558	99.550
19	99.542	99.534	99.526	99.518	99.510	99.501
20	99.493	99.485	99.476	99.467	99.459	99.450
21	99.441	99.432	99.423	99.414	99.405	99.396
22	99.387	99.378	99.368	99.359	99.349	99 340
23	99.330	99.320	99.310	93.301	99.291	99.281
24	99.271	99.260	99.250	99.240	99.230	99.219
25	99.208	99.198	99.187	99.177	99.166	99.155
26	99.144	99.133	99.122	99.111	99.100	99.088
27	99.077	99.066	99.054	99.043	99.031	99.020
28	99.008	98.996	98.984	98.972	98.960	98.948
29	98 936	98.924	98.911	98.898	98.887	98.874

Degree of Curve.		2 Sta.	3 Sta.	4 Sta.	5 Sta.	6 Sta.	7 Sta.	8 Sta.	9 Sta.	10 Sta.
0°	10′	200.00	300.00	400.00	499.99	599.99	699.99	799.98	899.97	999.96
	20	200.00	300.00	399.99	499.98	599.97	699.95	799.93	899.90	999.86
	30	200.00	299.99	399.98	499.96	599.93	699.89	799.84	899.77	999.68
	40	200.00	299.98	399.96	499.93	599.88	699.81	799.71	899.59	999.44
	50	199.99	299.98	399.94	499.89	599.81	699.70	799.55	899.36	999.12
1	00	199.99	299.97	399.92	499.84	599.73	699.56	799.35	899.08	998.73
	10	199.99	299.95	399.89	499.78	599.63	699.41	799.12	898.74	998.27
	20	199.98	299.94	399.86	499.72	599.51	699.23	798.85	898.36	997.75
	30	199.98	299.92	399.82	499.64	599.38	699.02	798.54	897.93	997.15
	40	199.97	299.90	399.77	499.56	599.24	698.79	798.20	897.43	996.48
	50	199.97	299.88	399.73	499.47	599.08	698.54	797.82	896.89	995.74
2	00	199.96	299.86	399.68	499.37	598.90	698.26	797.40	896.30	994.93
	10	199.95	299.84	399.62	499.26	598.71	697.96	796.96	895.66	994.05
	20	199.94	299.81	399.56	499.14	598.51	697.63	796.47	894.97	993.10
	30	199.94	299.79	399.49	499.01	598.29	697.28	795.94	894.23	992.09
	40	199.93	299.76	399.42	498.87	598.05	696.91	795.39	893.43	991.00
	50	199.92	299.72	399.35	498.73	597.80	696.51	794.78	892.59	989.84
3	00	199.91	299.69	399.27	498.57	597.54	696.09	794.16	891.70	988.62
	10	199.90	299.66	399.19	498.41	597.25	695.64	793.50	890.75	987.32
	20	199.89	299.62	399.10	498.24	596.96	695.17	792.80	889.75	985.96
	30	199.88	299.58	399.01	498.06	596.65	694.68	792.06	888.71	984.52
	40	199.86	299.54	398.91	497.87	596.32	694.16	791.29	887.61	983.02
	50	199.85	299.50	398.81	497.67	595.98	693.62	790.48	886.47	981.45
4	00	199.84	299.45	398.70	497.47	595.62	693.06	789.64	885.27	979.82
	10	199.82	299.41	398.59	497.25	595.25	692.47	788.77	884.02	978.11
	20	199.81	299.36	398.48	497.03	594.87	691.85	787.85	882.73	976.34
	30	199.79	299.31	398.36	496.79	594.46	691.22	786.91	881.38	974.50
	40	199.78	299.25	398.23	496.55	594.05	690.56	785.92	879.98	972.59
	50	199.76	299.20	398.11	496.30	593.62	689.87	784.90	878.54	970.61
5	00	199.75	299.14	397.97	496.04	593.17	689.17	783.85	877.05	968.57
	10	199.73	299.09	397.84	495.78	592.71	688.44	782.77	875.50	966.46
	20	199.71	299.03	397.69	495.50	592.23	687.68	781.64	873.91	964.29
	30	199.69	298.96	397.55	495.21	591.74	686.90	780.49	872.27	962.05
	40	199.67	298.90	397.40	494.92	591.24	686.10	779.30	870.58	959.74
	50	199.65	298.84	397.24	494.62	590.72	685.28	778.07	868.84	957.37
6	00	199.63	298.77	397.08	494.31	590.18	684.43	776.81	867.06	954.93
	10	199.61	298.70	396.92	493.99	589.63	683.56	775.52	865.22	952.43
	20	199.59	298.63	396.75	493.66	589.06	682.67	774.19	863.34	949.86
	30	199.57	298.55	396.58	493.32	588.48	681.75	772.83	861.41	947.25
	40	199.55	298.48	396.40	492.98	587.89	680.81	771.43	859.44	944.54
	50	199.53	298.40	396.22	492.62	587.28	679.85	770.00	857.41	941.78
7	00	199.50	298.32	396.03	492.26	586.66	678.86	768.54	855.34	938.96
	10	199.48	298.24	395.84	491.89	586.02	677.85	767.04	853.22	936.07
	20	199.45	298.16	395.65	491.51	585.36	676.82	765.51	851.06	933.13
	30	199.43	298.08	395.45	491.12	584.70	675.77	763.94	848.85	930.12
	40	199.40	297.99	395.24	490.73	584.02	674.69	762.35	846.59	927.05
	50	199.38	297.90	395.03	490.32	583.32	673.59	760.72	844.29	923.92
8	00	199.35	297.81	394.82	489.91	582.61	672.47	759.05	841.94	920.73
	10	199.32	297.72	394.60	489.49	581.88	671.32	757.36	839.55	917.47
	20	199.30	297.63	394.38	489.05	581.14	670.16	755.63	837.11	914.16
	30	199.27	297.53	394.16	488.62	580.39	668.97	753.87	834.62	910.79
	40	199.24	297.43	393.93	488.17	579.62	667.76	752.07	832.09	907.36
	50	199.21	297.33	393.69	487.71	578.84	666.52	750.23	829.52	903.87
9	00	199.18	297.23	393.45	487.25	578.04	665.27	748.39	826.90	900.32
	10	199.15	297.13	393.21	486.77	577.23	663.99	746.50	824.24	896.71
	20	199.12	297.02	392.96	486.29	576.40	662.69	744.58	821.54	893.04
	30	199.08	296.92	392.71	485.80	575.56	661.37	742.63	818.79	889.32
	40	199.05	296.81	392.45	485.31	574.71	660.02	740.64	816.00	885.55
	50	199.02	296.70	392.19	484.80	573.84	658.66	738.62	813.16	881.71
	00	198.99	296.58	391.93	484.28	572.96	657.27	736.58	810.28	877.82

Degree of Curve		2 Sta.	3 Sta.	4 Sta.	5 Sta.	6 Sta.	7 Sta.	8 Sta.	9 Sta.	10 Sta.
10°	10′	198.95	296.47	391.66	483.76	572.06	655.83	734.50	807.37	873.88
	20	198.92	296.35	391.38	483.23	571.15	654.43	732.39	804.40	869.88
	30	198.88	296.24	391.10	482.69	570.23	652.98	730.25	801.40	865.82
	40	198.85	296.12	390.82	482.14	569.29	651.51	728.08	798.36	861.72
	50	198.81	295.99	390.53	481.59	568.34	650.01	725.88	795.27	857.56
11	00	198.77	295.87	390.24	481.02	567.37	648.50	723.65	792.15	853.34
	10	198.74	295.75	389.95	480.45	566.39	646.96	721.39	788.98	849.08
	20	198.70	295.62	389.65	479.87	565.40	645.41	719.10	785.77	844.76
	30	198.66	295.49	389.34	479.28	564.39	643.83	716.78	782.53	840.40
	40	198.62	295.36	389.04	478.68	563.37	642.23	714.44	779.24	835.98
	50	198.58	295.22	388.72	478.08	562.34	640.61	712.06	775.92	831.51
12	00	198.54	295.09	388.40	477.46	561.29	638.92	709.65	772.55	826.99
	10	198.50	294.95	388.08	476.84	560.23	637.31	707.22	769.15	822.43
	20	198.46	294.81	387.76	476.21	559.16	635.63	704.75	765.71	817.81
	30	198.42	294.67	387.43	475.58	558.07	633.93	702.26	762.24	813.15
	40	198.37	294.53	387.09	474.93	556.97	632.21	699.74	758.72	808.44
	50	198.33	294.39	386.76	474.28	555.86	630.47	697.19	755.17	803.69
13	00	198.29	294.24	386.41	473.62	554.73	628.71	694.61	751.58	798.89
	10	198.24	294.09	386.07	472.95	553.59	626.93	692.01	747.95	794.04
	20	198.20	293.94	385.71	472.27	552.44	625.13	689.37	744.29	789.15
	30	198.15	293.79	385.36	471.58	551.27	623.31	686.71	740.60	784.21
	40	198.11	293.64	385.00	470.89	550.09	621.47	684.03	736.87	779.23
	50	198.06	293.49	384.64	470.19	548.90	619.62	681.32	733.10	774.21
14	00	198.02	293.33	384.27	469.48	547.69	617.74	678.58	729.30	769.15
	10	197.97	293.17	383.90	468.76	546.47	615.84	675.81	725.46	764.04
	20	197.92	293.01	383.52	468.04	545.24	613.93	673.02	721.60	758.90
	30	197.87	292.85	383.14	467.30	544.00	611.99	670.20	717.69	753.71
	40	197.82	292.68	382.75	466.56	542.74	610.04	667.36	713.76	748.48
	50	197.77	292.52	382.36	465.82	541.47	608.07	664.49	709.79	743.22
15	00	197.72	292.35	381.97	465.06	540.19	606.08	661.59	705.79	737.91
	10	197.67	292.18	381.57	464.30	538.90	604.07	658.68	701.76	732.57
	20	197.62	292.01	381.17	463.52	537.59	602.04	655.73	697.70	727.19
	30	197.57	291.83	380.77	462.74	536.27	600.00	652.76	693.61	721.78
	40	197.52	291.66	380.36	461.96	534.94	597.93	649.77	689.48	716.32
	50	197.46	291.48	379.94	461.16	533.59	595.85	646.75	685.33	710.84
16	00	197.41	291.30	379.53	460.36	532.24	593.75	643.71	681.14	705.32
	10	197.36	291.12	379.10	459.55	530.87	591.64	640.65	676.93	699.76
	20	197.30	290.94	378.68	458.74	529.49	589.50	637.56	672.69	694.17
	30	197.25	290.76	378.25	457.91	528.10	587.35	634.45	668.42	688.55
	40	197.19	290.57	377.81	457.08	526.69	585.18	631.32	664.12	682.91
	50	197.14	290.38	377.38	456.24	525.28	583.00	628.16	659.80	677.22
17	00	197.08	290.19	376.93	455.39	523.85	580.80	624.98	655.44	671.50
	10	197.02	290.00	376.49	454.54	522.41	578.58	621.78	651.06	665.76
	20	196.96	289.81	376.04	453.68	520.96	576.34	618.56	646.66	659.99
	30	196.90	289.61	375.58	452.81	519.49	573.99	615.32	642.23	654.19
	40	196.85	289.42	375.12	451.93	518.02	571.82	612.06	637.77	648.36
	50	196.79	289.22	374.66	451.05	516.53	569.53	608.77	633.29	642.50
18	00	196.73	289.02	374.20	450.16	515.04	567.23	605.46	628.78	636.42
	10	196.67	288.82	373.73	449.26	513.53	564.91	602.13	624.25	
	20	196.60	288.61	373.25	448.35	512.01	562.58	598.78	619.70	
	30	196.54	288.41	372.77	447.44	510.48	560.23	595.42	615.12	
	40	196.48	288.20	372.29	446.52	508.93	557.87	592.03	610.52	
	50	196.42	287.99	371.80	445.60	507.38	555.49	588.62	605.90	
19	00	196.36	287.78	371.33	444.66	505.83	553.09	585.22	601.25	
	10	196.29	287.57	370.82	443.72	504.24	550.68	581.75	596.59	
	20	196.23	287.35	370.32	442.77	502.65	548.26	578.29	591.90	
	30	196.16	287.14	369.82	441.82	501.05	545.81	574.81	587.20	
	40	196.10	286.92	369.31	440.86	499.44	543.36	571.31	582.47	
	50	196.03	286.70	368.80	439.89	497.83	540.89	567.79	577.72	
20	00	195.96	286.48	368.29	438.91	496.20	538.40	564.25	592.96	

Deg. of Curve.	2 Sta.	3 Sta.	4 Sta.	5 Sta.	6 Sta.	7 Sta.	8 Sta.	9 Sta.	10 Sta.
0° 10′	.145	.327	.582	.909	1.309	1.782	2.327	2.945	3.636
20	.291	.654	1.164	1.818	2.618	3.563	4.654	5.889	7.272
30	.436	.982	1.745	2.727	3.927	5.345	6.981	8.835	10.91
40	.582	1.309	2.327	3.636	5.235	7.126	9.807	11.78	14.54
50	.727	1.636	2.909	4.545	6.544	8.907	11.63	11.72	18.17
1 00	.873	1.963	3.490	5.453	7.852	10.69	13.96	17.66	21.80
10	1.018	2.291	4.072	6.362	9.160	12.47	16.28	20.60	25.43
20	1.164	2.618	4.653	7.270	10.47	14.25	18.60	23.54	29.06
30	1.309	2.945	5.235	8.178	11.77	16.02	20.92	26.48	32.68
40	1.454	3.272	5.816	9.086	13.08	17.80	23.24	29.41	36.30
50	1.600	3.599	6.397	9.994	14.89	19.58	25.56	32.34	39 91
2 00	1.745	3.926	6.978	10.90	15.69	21.35	27.88	35.27	43.52
10	1.891	4.253	7.560	11.81	17.00	23.13	30.19	38.20	47.13
20	2.036	4.580	8.140	12.72	18 30	24.90	32.51	41.12	50.73
30	2.181	4.907	8.721	13.62	19.61	26.67	34 82	44.04	54.33
40	2.329	5.234	9.302	14.53	20.91	28.44	37.13	46.95	57.92
50	2.472	5.561	9.882	15.43	22.21	30.21	39.43	49.86	61.50
3 00	2.617	5.887	10.46	16.34	23.51	31.98	41.73	52.77	65.08
10	2.763	6.214	11.04	17.24	24.81	33.75	44.04	55.67	68.65
20	2.908	6.541	11.62	18.15	26.11	35.51	46.33	58.57	72.21
30	3.053	6.868	12.20	19.05	27.41	37.27	48.63	61.46	75.77
40	3.199	7.194	12.78	19.96	28.71	39.06	50.92	64.35	79.31
50	3.344	7.520	13.36	20.86	30.01	40.79	53.20	67.23	82.85
4 00	3.489	7.847	13.94	21.76	31.30	42.55	55.49	70.11	86.38
10	3.635	8.173	14.52	22.66	32.60	44.30	57.77	72.98	89.91
20	3.780	8 499	15.10	23.56	33.89	46.05	60.04	75.84	93.42
30	3.925	8.826	15.68	24.46	35.18	47.80	62.32	78.70	96.92
40	4.070	9.152	16.25	25.36	36.47	49.55	64.58	81.55	100.41
50	4.215	9.478	16.83	26.26	37.76	51.29	66.85	84.39	103.89
5 00	4.361	9.803	17.41	27.16	39.05	53.04	69.11	87.28	107.36
10	4.506	10.13	17.99	28.06	40.33	54.78	71.36	90.06	110.82
20	4.651	10.46	18.56	28.96	41.62	56.51	73.61	92.88	114.27
30	4.796	10.78	19.14	29.85	42.90	58.24	75.86	95.69	117.71
40	4.941	11.11	19.72	30.75	44.18	59.97	78.10	98.50	121.13
50	5.086	11.43	20 29	31.64	45.46	61.70	80.33	101.29	124.54
6 00	5.231	11.76	20.87	32.54	46.74	63.48	82.56	104.08	127.94
10	5.376	12.08	21.44	33.43	48.01	65.15	84.78	106.86	131.32
20	5.521	12.41	22.02	34.32	49.29	66.86	87.00	109.63	134.69
30.	5.666	12.73	22.59	35.22	50.56	68.58	89.21	112.39	138.05
40	5.811	13.06	23.17	36.11	51.83	70.29	91.42	115.14	141.39
50	5.956	13.38	23.74	36.99	53.10	71.99	93.62	117.88	144.71
7 00	6.101	13.71	24.31	37.88	54.37	73.70	95.81	120.62	148.03
10	6.246	14.03	24.89	38.77	55.63	75.40	97.99	123.34	151.32
20	6.391	14.35	25.46	39.66	56.89	77.09	100.18	126.05	154.60
30	6.536	14.68	26.03	40.54	58.15	78.78	102.35	128.75	157.87
40	6.680	15.00	26.60	41.43	59 41	80.47	104.52	131.44	161.11
50	6.825	15.33	27.17	42.31	60.67	82.15	106.67	134.11	164.34
8 00	6.970	15.65	27.74	43.19	61.92	83.83	108.83	136.78	167.56
10	7.115	15.97	28.31	44.07	63.17	85.51	110.97	139.44	170.75
20	7.259	16.30	28.88	44.95	64.42	87.18	113.11	142.08	173.93
50	7.404	16.62	29.45	45.83	65.66	88.84	115.24	144.71	177.09
40	7.549	16.94	30.02	46.71	66.91	90.50	117.36	147.33	180.23
50	7.693	17.27	30.59	47.58	68.15	92.16	119.48	149.94	183.36
9 00	7.888	17.59	31.16	48.46	69.39	93.81	121.58	152.53	186.46
10	7.982	17.91	31.73	49.33	70.62	95.46	123.68	155.11	189.55
20	8.127	18.23	32.29	50.21	71.86	97.10	125.77	157.68	192.61
30	8.271	18.56	32.86	51.08	73 09	98.74	127.85	160.23	195.66
40	8.416	18.88	33.42	51.95	74.31	100.37	129.93	162.77	198.68
50	8.560	19.20	33.99	52.81	75.54	102.00	131.99	165.30	201.68
10 00	8.705	19.52	34.55	53.68	76.76	103.62	134.05	167.82	204.67

Deg. of Curve.	2 Sta.	3 Sta.	4 Sta.	5 Sta.	6 Sta.	7 Sta.	8 Sta.	9 Sta.	10 Sta.
10° 10′	8.849	19.84	35.12	54.55	77.98	105.23	136.09	170.31	207.63
20	8.993	20.17	35.68	55.41	79.20	106.85	138.13	172.80	210.57
30	9.137	20.49	36.24	56.27	80.41	108.45	140.16	175.27	213.49
40	9.282	20.81	36.81	57.14	81.62	110.05	142.18	177.73	216.39
50	9.426	21.13	37.37	57.99	82.83	111.64	144.19	180.17	219.26
11 00	9.570	21.45	37.93	58.85	84.03	113.28	146.19	182.59	222.11
10	9.714	21.77	38.49	59.71	85.23	114.82	148.18	185.00	224.94
20	9.858	22.09	39.05	60.56	86.43	116.39	150.16	187.40	227.75
30	10.00	22.41	39.61	61.42	87.62	117.96	152.13	189.78	230.53
40	10.15	22.73	40.16	62.27	88.82	119.53	154.09	192.14	233.29
50	10.29	23.05	40 72	63.12	90.00	121.09	156.04	194.49	236.02
12 00	10.43	23.37	41.28	63.97	91.19	122.64	157.98	196.82	238.73
10	10.58	23.69	41.84	64.81	92.37	124.19	159.91		
20	10.72	24.01	42.39	65.66	93.55	125.78	161.83		
30	10.86	24.33	42.95	66.50	94.72	127.26	163.73		
40	11.01	24.64	43.50	67.35	95.89	128.79	165.63		
50	11.15	24.96	44.05	68.19	97.06	130.31	167.52		
13 00	11.30	25.28	44.61	69.02	98.22	131.82	169.39		
10	11.44	25.60	45.16	69.86	99.38	133.33	171.26		
20	11.58	25.92	45.71	70.69	100.54	134.83	173.11		
30	11.73	26.23	46.26	71.53	101.69	136.32	174.95		
40	11.87	26.55	46.81	72.36	102.83	137.81	176.78		
50	12.01	26.87	47.36	73.19	103.98	139.29	178.60		
14 00	12.16	27.18	47.90	74.01	105.12	140.76	180.40		
10	12.30	27.50	48.45	74.84	106.26	142.23			
20	12.44	27.81	49.00	75.66	107.39	143.68			
30	12.59	28.13	49.54	76.48	108.52	145.13			
40	12.73	28.45	50.09	77.30	109.64	146.58			
50	12.87	28.76	50.63	78.12	110.76	148.01			
15 00	13.02	29 08	51.17	78.93	111.88	149.44			
10	13.16	29.39	51.72	79.75	112.99	150.86			
20	13.30	29.70	52.26	80.56	114.10	152.28			
30	13.44	30.02	52.80	81.37	115.20	153.68			
40	13.59	30.33	53.34	82.17	116.30	155 08			
50	13.73	30.65	53.88	82.98	117.39	156.47			
16 00	13.87	30.96	54.41	83.78	118.48	157.85			
10	14 01	31.27	54.95	84.58	119 57				
20	14.16	31.58	55.49	85.38	120.65				
30	14.30	31.90	56.02	86.17	121.73				
40	14.44	32.21	56.55	86.97	122.80				
50	14.58	32.52	57.09	87.76	123.87				
17 00	14.73	32.83	57.62	88.55	124.93				
10	14.87	33.14	58.15	89 33	125.99				
20	15.01	33.45	58.68	90.12	127.04				
30	15.15	33.76	59.21	90.90	128.09				
40	15.30	34.07	59.74	91.68	129.14				
50	15.44	34.38	60.27	92.46	130.18				
18 00	15.58	34.69	60.79	93.23	131.21				
10	15.72	35.00	61.32	94.00	132.24				
20	15.86	35.31	61.84	94.77	133.27				
30	16.00	35.62	62.36	95.54	134.29				
40	16.15	35.93	62.89	96.31	135.30				
50	16.29	36.24	63.41	97.07	136.31				
19 00	16.43	36.54	63.93	97.83	137.32				
10	16.57	36.85	64.44	98.59	138.32				
20	16.71	37.16	64.96	99.34	139.31				
30	16.85	37.46	65.48	100.09	140.30				
40	16.99	37.77	66.00	100.84	141.29				
50	17.14	38.08	66.51	101.59	142.27				
20 00	17.28	38.38	67.02	102.33	143.24				

Switch-rail 15 Feet, thrown 5 inches.

No. *n*.	Angle *F*.	Radius, Center.	Degree of Curve.	Lead *HF*.	No. *n″*.	Angle *F″*.	Dist. *H′F″*.	*LF*.
	° ′		° ′ ″			° ′		
4	14 15	138.89	41 15 09	45.84	2.95	19 16	34.71	34.77
4½	12 41	176.42	32 28 30	49.28	3.31	17 11	36.74	39.24
5	11 25	218 09	26 11 59	52.63	3.66	15 33	38.64	43.77
5½	10 23	265.75	21 33 36	55.90	4 03	14 09	40.57	48.35
6	9 32	317.68	18 21 25	59.08	4.38	13 01	42.37	52.97
6½	8 48	374.75	15 17 21	62.20	4.74	12 03	44.11	57.68
7	8 10	437 06	13 06 34	65.25	5.09	11 14	45.78	62.46
7½	7 38	504.69	11 21 08	68.22	5.43	10 31	47.38	67.37
8	7 09	577.85	9 54 55	71.12	5.78	9 54	48.92	72.24
8½	6 44	656.79	8 43 24	73.98	6.12	9 21	50.40	77.28
9	6 22	741.67	7 43 50	76.75	6.45	8 52	51.80	82.49
9½	6 02	832.75	6 52 45	79.47	6.78	8 26	53.20	87.66
10	5 43	930.42	6 09 29	82.12	7.11	8 03	54.52	93.03
10½	5 27	1034.56	5 32 19	84.73	7.43	7 42	55.78	98.53
11	5 12	1145.90	5 00 60	87.27	7.75	7 23	57.01	104.17
11½	4 59	1264.40	4 31 53	88.76	8.07	7 06	58.18	109.95
12	4 46	1390.90	4 07 10	92.20	8.37	6 50	59.30	115.96

Switch-rail 8 Feet, thrown 5 inches.

No. *n*.	Angle *F*.	Radius, Center.	Degree of Curve.	Lead *HF*.	No. *n″*.	Angle *F″*.	Dist. *H′F*.	*LF*.
	° ′ ″		° ′ ″			° ′		
3	18 55 28	78.08	72 27 11	30.17	2.19	25 45	21.91	26.41
3½	16 15 32	108.73	52 41 44	33.31	2.80	22 15	23.65	31.10
4	14 15 00	143.56	39 54 26	36.32	2.89	19 38	25.27	35.92
4½	12 40 48	183.98	31 03 48	39.20	3.23	17 35	26.78	40.90
5	11 25 16	230.31	24 52 32	41.96	3.57	15 58	28.18	46.07
5½	10 23 20	283.05	20 14 26	44.60	3.89	14 38	29.49	51.47
6	9 31 39	342.70	16 43 29	47.15	4.21	13 32	30.70	57.12
6½	8 47 51	409.97	13 58 28	49.59	4.52	12 37	31.84	63 07

Angle *V.*	Tang. *T.*	Ex. *E.*	Angle *V.*	Tang. *T.*	Ex. *E.*	Angle *V.*	Tang. *T.*	Ex. *E.*
90° 10′	344776	143105	**100°** 10′	410907	191973	**110°** 10′	492484	256826
20	345781	143817	20	412123	192907	20	494013	258081
30	346788	144531	30	413324	193845	30	495549	259342
40	347798	145249	40	414569	194787	40	497091	260610
50	348811	145971	50	415798	195724	50	498767	261884
91 00	349828	146695	**101** 00	417032	196685	**111** 00	500195	263165
10	350847	147422	10	418270	197641	10	501757	264453
20	351869	148153	20	419512	198601	20	503326	265748
30	352895	148887	30	420759	199566	30	504901	267049
40	353923	149624	40	422010	200536	40	506483	268357
50	354955	150365	50	423266	201510	50	508071	269673
92 00	355989	151108	**102** 00	424526	202489	**112** 00	509667	270995
10	357027	151855	10	425791	203472	10	511269	272324
20	358068	152606	20	427060	204460	20	512879	273660
30	359112	153359	30	428334	205453	30	514495	275003
40	360159	154116	40	429613	206451	40	516119	276354
50	361209	154877	50	430896	207453	50	517749	277712
93 00	362263	155640	**103** 00	432184	208461	**113** 00	519387	279077
10	363320	156408	10	433477	209473	10	521032	280449
20	364380	157178	20	434774	210490	20	522684	281829
30	365443	157952	30	436076	211512	30	524344	283216
40	366510	158730	40	437383	212539	40	526010	284611
50	367580	159511	50	438695	213571	50	527685	286013
94 00	368653	160295	**104** 00	440012	214608	**114** 00	529367	287423
10	369730	161083	10	441338	215650	10	531056	288840
20	370810	161874	20	442660	216697	20	532753	290265
30	371893	162670	30	443991	217749	30	534457	291698
40	372980	163468	40	445328	218807	40	536170	293139
50	374070	164270	50	446669	219869	50	537890	294588
95 00	375164	165076	**105** 00	448016	220937	**115** 00	539618	296045
10	376261	165886	10	449368	222010	10	541354	297509
20	377362	166699	20	450725	223088	20	543098	299051
30	378466	167516	30	452087	224172	30	544850	300463
40	379574	168336	40	453454	225261	40	546610	301952
50	380686	169161	50	454827	226355	50	548378	303450
96 00	381800	169989	**106** 00	456204	227455	**116** 00	550155	304956
10	382919	170820	10	457588	228560	10	551939	306472
20	384041	171656	20	458976	229671	20	553732	307992
30	385167	172495	30	460370	230788	30	555534	309522
40	386297	173338	40	461770	232909	40	557344	311063
50	387430	174186	50	463174	233037	50	559162	312612
97 00	388567	175036	**107** 00	464585	234170	**117** 00	560989	314169
10	389707	175891	10	466001	235309	10	562825	315735
20	390852	176750	20	467422	236453	20	564670	317310
30	392000	177613	30	468850	237604	30	566523	318894
40	393152	178480	40	470282	238760	40	568386	320487
50	394308	179350	50	471721	239922	50	570257	322089
98 00	395468	180225	**108** 00	473165	241090	**118** 00	572137	323700
10	396631	181104	10	474615	242264	10	574027	325320
20	397799	181987	20	476071	243443	20	575925	326950
30	398970	182873	30	477533	244629	30	577833	328589
40	400146	183764	40	479001	245821	40	579751	330237
50	401325	184660	50	480475	247019	50	581677	331895
99 00	402508	185559	**109** 00	481954	248223	**119** 00	583614	333563
10	403696	186462	10	483440	249433	10	585559	335240
20	404887	187370	20	484932	250649	20	587515	336927
30	406083	188282	30	486430	251872	30	589480	338624
40	407283	189198	40	487934	253101	40	591455	340331
50	408487	190119	50	489444	254336	50	593440	342048
100 00	409695	191044	**110** 00	490961	255578	**120** 00	595435	343775

Angle *V*.	Tang. *T*.	Ext. *E*.	Angle *V*.	Tang. *T*.	Ext. *E*.	Angle *V*.	Tang. *T*.	Ext. *E*.
60° 10′	199146	53516.2	**70°** 10′	241460	76325.0	**80°** 10′	289314	105540
20	199814	53851.6	20	242207	76755.0	20	290169	106091
30	200483	54188.4	30	242956	77186.8	30	291027	106645
40	201151	54526.7	40	243706	77620.4	40	291886	107200
50	201826	54866.4	50	244458	78055.8	50	292748	107759
61 00	202499	55207.5	**71** 00	245212	78493.0	**81** 00	293611	108319
10	203173	55550.0	10	245967	78932.0	10	294477	108882
20	203848	55891.0	20	246724	79372.8	20	295345	109447
30	204524	56239.4	30	247482	79815.4	30	296215	110014
40	205202	56586.3	40	248242	80259.8	40	297088	110584
50	205881	56934.6	50	249004	80706.1	50	297962	111157
62 00	206561	57284.3	**72** 00	249767	81154.2	**82** 00	298839	111731
10	207242	57635.6	10	250532	81604.1	10	299718	112308
20	207924	57988.3	20	251298	82056.0	20	300599	112888
30	208608	58342.4	30	252066	82509.6	30	301482	113470
40	209292	58698.1	40	252836	82965.2	40	302368	114054
50	209978	59055.2	50	253607	83422.6	50	303256	114641
63 00	210665	59413.8	**73** 00	254380	83877.5	**83** 00	304146	115231
10	211354	59773.9	10	255154	84343.1	10	305039	115823
20	212043	60135.5	20	255931	84806.2	20	305933	116417
30	212734	60498.6	30	256709	85271.3	30	306831	117014
40	213426	60863.2	40	257488	85738.2	40	307730	117613
50	214120	61229.3	50	258270	86207.0	50	308632	118215
64 00	214814	61597.0	**74** 00	259053	86677.9	**84** 00	309536	118820
10	215510	61966.2	10	259838	87150.6	10	310443	119427
20	216207	62336.9	20	260624	87625.3	20	311352	120037
30	216906	62709.1	30	261412	88102.0	30	312263	120649
40	217605	63082.9	40	262202	88580.6	40	313177	121264
50	218306	63458.3	50	262994	89061.2	50	314093	121881
65 00	219009	63835.2	**75** 00	263788	89543.8	**85** 00	315011	122501
10	219712	64213.7	10	264583	90028.4	10	315933	123124
20	220417	64593.7	20	265380	90515.0	20	316856	123749
30	221123	64975.3	30	266179	91003.6	30	317782	124378
40	221831	65358.5	40	266979	91494.3	40	318711	125008
50	222540	65743.3	50	267782	91987.0	50	319642	125642
66 00	223250	66129.6	**76** 00	268586	92481.7	**86** 00	320575	126278
10	223961	66517.6	10	269392	92978.4	10	321511	126917
20	224674	66907.2	20	270200	93477.2	20	322450	127558
30	225389	67298.3	30	271010	93978.1	30	323391	128203
40	226104	67691.2	40	271822	94481.1	40	324335	128850
50	226821	68085.6	50	272635	94986.1	50	325281	129500
67 00	227540	68481.6	**77** 00	273451	95493.3	**87** 00	326230	130153
10	228259	68879.8	10	274268	96002.5	10	327182	130808
20	228980	69278.6	20	275087	96513.9	20	328136	131466
30	229703	69679.6	30	275908	97027.4	30	329093	132128
40	230427	70082.3	40	276731	97543.0	40	330052	132792
50	231152	70486.5	50	277556	98060.8	50	331014	133459
68 00	231879	70892.5	**78** 00	278383	98580.7	**88** 00	331979	134128
10	232607	71300.1	10	279212	99102.8	10	332947	134801
20	233337	71709.5	20	280043	99627.0	20	333917	135477
30	234068	72120.5	30	280876	100153	30	334890	136155
40	234800	72533.2	40	281710	100682	40	335866	136837
50	235584	72947.6	50	282547	101213	50	336845	137521
69 00	236270	73363.7	**79** 00	283386	101746	**89** 00	337826	138208
10	237007	73781.5	10	284227	102281	10	338811	138899
20	237745	74201.1	20	285070	102819	20	339798	139592
30	238485	74622.4	30	285914	103358	30	340788	140289
40	239226	75045.4	40	286761	103901	40	341780	140988
50	239969	75470.2	50	287610	104445	50	342776	141691
70 00	240714	75896.7	**80** 00	288461	104991	**90** 00	343775	142396

Angle *V*.	Tang. *T*.	Ex. *E*.	Angle *V*.	Tang. *T*.	Ex. *E*.	Angle *V*.	Tang. *T*.	Ex. *E*.
0° 10′	500.0	.364	**10°** 10′	30580.3	1357.4	**20°** 10′	61132.4	5393.2
20	1000.0	1.454	20	31084.3	1402.5	20	61648.4	5483.9
30	1500.0	3.273	30	31588.5	1448.2	30	62164.6	5575.4
40	2000.0	5.818	40	32092.7	1494.7	40	62681.1	5667.6
50	2500.0	9.090	50	32597.2	1542.0	50	63197.8	5760.7
1 00	3000.1	13.090	**11** 00	33101.7	1590.0	**21** 00	63714.9	5854.6
10	3500.1	17.818	10	33606.5	1638.7	10	64232.2	5949.2
20	4000.2	23.272	20	34111.3	1688.2	20	64749.8	6044.7
30	4500.3	29.455	30	34616.3	1738.4	30	65267.7	6140.9
40	5000.4	36.364	40	35121.4	1789.4	40	65785.8	6237.9
50	5500.5	44.002	50	35626.7	1841.2	50	66304.3	6335.7
2 00	6000.6	52.367	**12** 00	36132.2	1893.6	**22** 00	66823.0	6434.3
10	6500.8	61.459	10	36637.8	1946.8	10	67342.1	6533.7
20	7001.0	71.280	20	37143.5	2000.8	20	67861.4	6633.9
30	7501.2	81.829	30	37649.4	2055.5	30	68381.1	6735.0
40	8001.4	93.105	40	38155.5	2110.9	40	68901.0	6836.8
50	8501.7	105.11	50	38661.8	2167.2	50	69421.2	6939.4
3 00	9002.1	117.84	**13** 00	39168.2	2224.1	**23** 00	69941.8	7042.8
10	9502.4	131.31	10	39674.8	2281.8	10	70462.6	7147.0
20	10002.8	145.50	20	40181.5	2340.3	20	70983.8	7252.0
30	10503.3	160.41	30	40688.4	2399.5	30	71505.2	7357.8
40	11003.8	176.06	40	41195.5	2459.5	40	72027.3	7464.4
50	11504.3	192.44	50	41702.8	2520.2	50	72549.1	7571.9
4 00	12004.9	209.55	**14** 00	42210.2	2581.7	**24** 00	73071.6	7680.1
10	12505.5	227.38	10	42717.9	2643.9	10	73594.3	7789.2
20	13006.2	245.95	20	43225.7	2706.9	20	74117.4	7899.0
30	13506.9	265.24	30	43733.7	2770.7	30	74640.8	8009.7
40	14007.7	285.27	40	44241.8	2835.1	40	75164.6	8121.2
50	14508.6	306.02	50	44750.2	2900.4	50	75688.6	8233.5
5 00	15009.5	327.51	**15** 00	45258.8	2966.4	**25** 00	76213.0	8346.7
10	15510.5	349.72	10	45767.6	3033.2	10	76737.8	8460.6
20	16011.6	372.67	20	46276.5	3101.7	20	77262.8	8575.4
30	16512.7	396.35	30	46785.7	3169.0	30	77788.3	8691.0
40	17013.9	420.76	40	47295.0	3238.1	40	78314.0	8807.4
50	17515.1	445.90	50	47804.6	3307.9	50	78840.2	8924.6
6 00	18016.5	471.78	**16** 00	48314.4	3378.5	**26** 00	79366.6	9042.7
10	18517.9	498.38	10	48824.4	3449.8	10	79893.5	9161.6
20	19019.4	525.72	20	49334.6	3521.9	20	80420.7	9281.3
30	19520.9	553.79	30	49845.0	3594.8	30	80948.2	9401.8
40	20022.6	582.60	40	50355.6	3668.4	40	81476.1	9523.2
50	20524.3	612.13	50	50866.4	3742.8	50	82004.4	9645.4
7 00	21026.2	642.41	**17** 00	51377.5	3818.0	**27** 00	82533.0	9768.4
10	21528.1	673.41	10	51888.8	3893.9	10	83062.0	9892.3
20	22030.1	705.15	20	52400.3	3970.7	20	83591.4	10017.0
30	22532.2	737.62	30	52912.0	4048.1	30	84121.1	10142.5
40	23034.4	770.83	40	53424.0	4126.4	40	84651.3	10268.9
50	23536.7	804.78	50	53936.2	4205.4	50	85181.8	10396.1
8 00	24039.1	839.46	**18** 00	54448.5	4285.2	**28** 00	85712.7	10524.2
10	24541.6	874.88	10	54961.2	4365.8	10	86243.9	10653.1
20	25044.2	911.03	20	55474.1	4447.1	20	86775.6	10782.8
30	25546.9	947.92	30	55987.3	4529.2	30	87307.6	10913.4
40	26049.7	985.55	40	56500.6	4612.1	40	87840.1	11044.9
50	26552.6	1023.91	50	57014.3	4695.8	50	88372.9	11177.1
9 00	27055.7	1063.91	**19** 00	57528.2	4780.2	**29** 00	88906.2	11310.3
10	27558.8	1102.86	10	58042.8	4865.4	10	89439.8	11444.3
20	28062.1	1143.43	20	58556.7	4951.5	20	89973.9	11579.1
30	28565.5	1184.75	30	59071.3	5038.2	30	90508.3	11714.8
40	29069.0	1226.83	40	59586.2	5125.8	40	91043.2	11851.4
50	29572.6	1269.63	50	60101.3	5214.1	50	91578.5	11988.8
10 00	30076.4	1313.15	**20** 00	60616.8	5303.3	**30** 00	92114.2	12127.1

Angle *V.*	Tang. *T.*	Ext. *E.*	Angle *V.*	Tang. *T.*	Ext. *E.*	Angle *V.*	Tang. *T.*	Ext. *E.*
30° 10′	92650.3	12266.2	**40°** 10′	125690	22256.8	**50°** 10′	160914	35796.5
20	93186.8	12406.2	20	126257	22452.0	20	161524	36055.5
30	93723.7	12547.1	30	126825	22648.1	30	162135	36315.7
40	94261.1	12688.8	40	127394	22845.2	40	162746	36577.0
50	94798.9	12831.4	50	127963	23043.3	50	163359	36839.4
31 00	95337.1	12974.8	**41** 00	128532	23242.4	**51** 00	163972	37103.1
10	95880.5	13119.2	10	129102	23442.5	10	164586	37367.9
20	96414.9	13264.3	20	129673	23643.6	20	165201	37633.9
30	96954.5	13410.4	30	130245	23845.6	30	165817	37901.1
40	97494.5	13557.4	40	130817	24048.7	40	166434	38169.4
50	98034.9	13705.2	50	131389	24252.8	50	167052	38439.0
32 00	98575.8	13853.9	**42** 00	131963	24457.9	**52** 00	167670	38709.7
10	99117.2	14003.5	10	132537	24663.9	10	168290	38981.6
20	99658.9	14154.0	20	133111	24871.0	20	168910	39254.7
30	100201	14305.4	30	133687	25079.2	30	169531	39529.1
40	100744	14457.6	40	134263	25288.3	40	170153	39804.6
50	101287	14610.7	50	134839	25498.5	50	170776	40081.3
33 00	101831	14764.7	**43** 00	135416	25709.6	**53** 00	171400	40359.3
10	102375	14919.7	10	135994	25921.8	10	172024	40638.4
20	102919	15075.5	20	136573	26135.0	20	172650	40918.8
30	103464	15232.2	30	137152	26349.3	30	173277	41200.4
40	104010	15389.8	40	137732	26564.6	40	173904	41483.3
50	104556	15548.3	50	138313	26780.9	50	174533	41767.4
34 00	105102	15707.7	**44** 00	138894	26998.2	**54** 00	175162	42052.7
10	105649	15868.0	10	139476	27216.6	10	175792	42339.2
20	106197	16029.2	20	140059	27436.1	20	176423	42627.0
30	106745	16191.3	30	140642	27656.6	30	177056	42916.0
40	107293	16354.3	40	141226	27878.1	40	177689	43206.3
50	107842	16518.2	50	141811	28100.7	50	178323	43497.9
35 00	108392	16683.1	**45** 00	142396	28324.4	**55** 00	178958	43790.7
10	108942	16848.8	10	142982	28549.0	10	179594	44084.8
20	109492	17015.5	20	143569	28774.8	20	180231	44380.1
30	110043	17183.1	30	144157	29001.6	30	180869	44676.7
40	110595	17351.6	40	144745	29229.5	40	181508	44974.6
50	111147	17521.1	50	145334	29458.5	50	182147	45273.8
36 00	111699	17691.4	**46** 00	145924	29688.5	**56** 00	182788	45574.2
10	112252	17862.7	10	146514	29919.6	10	183430	45876.0
20	112806	18034.9	20	147105	30151.8	20	184073	46179.0
30	113360	18208.1	30	147697	30385.8	30	184717	46483.4
40	113915	18382.1	40	148290	30619.4	40	185362	46789.0
50	114470	18557.1	50	148883	30854.8	50	186008	47095.9
37 00	115025	18733.1	**47** 00	149478	31091.4	**57** 00	186654	47404.2
10	115582	18909.9	10	150072	31329.0	10	187302	47713.8
20	116138	19087.8	20	150668	31567.7	20	187951	48024.6
30	116696	19266.5	30	151264	31807.5	30	188601	48336.9
40	117254	19446.3	40	151862	32048.4	40	189252	48650.4
50	117812	19626.9	50	152460	32290.4	50	189905	48965.3
38 00	118371	19808.5	**48** 00	153058	32533.6	**58** 00	190557	49281.5
10	118931	19991.1	10	153658	32777.8	10	191212	49599.1
20	119491	20174.6	20	154258	33023.1	20	191867	49918.0
30	120051	20359.1	30	154859	33269.6	30	192523	50238.2
40	120613	20544.5	40	155461	33517.2	40	193180	50559.8
50	121175	20730.9	50	156064	33765.9	50	193839	50882.8
39 00	121737	20918.2	**49** 00	156667	34015.8	**59** 00	194498	51207.1
10	122300	21106.6	10	157271	34266.7	10	195159	51532.9
20	122864	21295.8	20	157876	34518.9	20	195821	51859.9
30	123428	21486.1	30	158482	34772.1	30	196483	52188.4
40	123992	21677.3	40	159089	35026.5	40	197147	52518.3
50	124558	21869.5	50	159696	35282.0	50	197812	52849.5
40 00	125124	22062.7	**50** 00	160304	35538.7	**60** 00	198478	53182.1

TABLE VIII.—LENGTHS OF ARCS CORRESPONDING TO ANY NUMBER OF DEGREES, MINUTES, OR SECONDS FOR RADIUS = 1.

No. of Deg., Min., or Sec.	Arc for Degrees.	Arc for Minutes.	Arc for Seconds.	No. of Deg., Min., or Sec.	Arc for Degrees.	Arc for Minutes.	Arc for Seconds.
1	.0174533	.0002909	.0000048	31	.5410521	.0090175	.0001503
2	.0349066	.0005818	.0000097	32	.5585054	.0093084	.0001551
3	.0523599	.0008727	.0000145	33	.5759587	.0095993	.0001600
4	.0698132	.0011636	.0000194	34	.5934119	.0098902	.0001648
5	.0872665	.0014544	.0000242	35	.6108652	.0101811	.0001697
6	.1047198	.0017453	.0000291	36	.6283185	.0104720	.0001745
7	.1221730	.0020362	.0000339	37	.6457718	.0107629	.0001794
8	.1396263	.0023271	.0000388	38	.6632251	.0110538	.0001842
9	.1570796	.0026180	.0000436	39	.6806784	.0113446	.0001891
10	.1745329	.0029089	.0000485	40	.6981317	.0116355	.0001939
11	.1919862	.0031998	.0000533	41	.7155850	.0119264	.0001988
12	.2094395	.0034907	.0000582	42	.7330383	.0122173	.0002036
13	.2268928	.0037815	.0000630	43	.7504916	.0125082	.0002085
14	.2443461	.0040724	.0000679	44	.7679449	.0127991	.0002133
15	.2617994	.0043633	.0000727	45	.7853982	.0130900	.0002182
16	.2792527	.0046542	.0000776	46	.8028515	.0133809	.0002230
17	.2967060	.0049451	.0000824	47	.8203047	.0136717	.0002279
18	.3141593	.0052360	.0000873	48	.8377580	.0139626	.0002327
19	.3316126	.0055269	.0000921	49	.8552113	.0142535	.0002376
20	.3490659	.0058178	.0000970	50	.8726646	.0145444	.0002424
21	.3665191	.0061087	.0001018	51	.8901179	.0148353	.0002473
22	.3839724	.0063995	.0001067	52	.9075712	.0151262	.0002521
23	.4014257	.0066904	.0001115	53	.9250245	.0154171	.0002570
24	.4188790	.0069813	.0001164	54	.9424778	.0157080	.0002618
25	.4363323	.0072722	.0001212	55	.9599311	.0159989	.0002666
26	.4537856	.0075631	.0001261	56	.9773844	.0162897	.0002715
27	.4712389	.0078540	.0001301	57	.9948377	.0165806	.0002763
28	.4886922	.0081449	.0001357	58	1.0122910	.0168715	.0002812
29	.5061455	.0084358	.0001406	59	1.0297443	.0171624	.0002860
30	.5235988	.0087266	.0001454	60	1.0471976	.0174533	.0002909

TABLE IX.—ACRES FOR VARIOUS LENGTHS AND WIDTHS.

Lengths.	Widths.								
	100	90	80	70	60	50	40	30	20
100	.229568	.206612	.183655	.160698	.137741	.114784	.091827	.068871	.045914
200	.459137	.413223	.367309	.321396	.275482	.229568	.183655	.137741	.091827
300	.688705	.619835	.550964	.482094	.413223	.344353	.275482	.206612	.137741
400	.918274	.826446	.734619	.642792	.550964	.459137	.367309	.275482	.183655
500	1.147842	1.033058	.918274	.803489	.688705	.573921	.459137	.344353	.229568
600	1.377410	1.239669	1.101928	.964187	.826446	.688705	.550964	.413223	.275482
700	1.606979	1.446281	1.285583	1.124885	.964187	.803489	.642792	.482094	.321396
800	1.836547	1.652893	1.469238	1.285583	1.101928	.918274	.734619	.550964	.367309
900	2.066116	1.859504	1.652892	1.446281	1.239669	1.033058	.826446	.619835	.413223

Sta.	Grade per Station.								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
3	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7
4	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6
5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
6	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
8	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
10	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
13	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
14	1.4	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6
15	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4
17	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
18	1.8	3.6	5.4	7.2	9.0	10.8	12.6	14.4	16.2
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1
20	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
21	2.1	4.2	6.3	8.4	10.5	12.6	14.7	16.8	18.9
22	2.2	4.4	6.6	8.8	11.0	13.2	15.4	17.6	19.8
23	2.3	4.6	6.9	9.2	11.5	13.8	16.1	18.4	20.7
24	2.4	4.8	7.2	9.6	12.0	14.4	16.8	19.2	21.6
25	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5
26	2.6	5.2	7.8	10.4	13.0	15.6	18.2	20.8	23.4
27	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
28	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1
30	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0
31	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9
32	3.2	6.4	9.6	12.8	16.0	19.2	22.4	25.6	28.8
33	3.3	6.6	9.9	13.2	16.5	19.8	23.1	26.4	29.7
34	3.4	6.8	10.2	13.6	17.0	20.4	23.8	27.2	30.6
35	3.5	7.0	10.5	14.0	17.5	21.0	24.5	28.0	31.5
36	3.6	7.2	10.8	14.4	18.0	21.6	25.2	28.8	32.4
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3
38	3.8	7.6	11.4	15.2	19.0	22.8	26.6	30.4	34.2
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40.5
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0

TABLE XI.—CORRECTION FOR THE EARTH'S CURVATURE AND FOR REFRACTION.

Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*
300	.002	1800	.066	3300	.223	4800	.472
400	.003	1900	.074	3400	.237	4900	.492
500	.005	2000	.082	3500	.251	5000	.512
600	.007	2100	.090	3600	.266	5100	.533
700	.010	2200	.099	3700	.281	5200	.554
800	.013	2300	.108	3800	.296	1 mile	.571
900	.017	2400	.118	3900	.312	2 miles	2.285
1000	.020	2500	.128	4000	.328	3 "	5.142
1100	.025	2600	.139	4100	.345	4 "	9.142
1200	.030	2700	.149	4200	.362	5 "	14.284
1300	.035	2800	.161	4300	.379	6 "	20.568
1400	.040	2900	.172	4400	.397	7 "	27.996
1500	.046	3000	.184	4500	.415	8 "	36.566
1600	.052	8100	.197	4600	.434	9 "	46.279
1700	.059	8200	.210	4700	.453	10 "	57.135

TABLE XII.—ELEVATION OF OUTER RAIL ON CURVES.

Degree *D.*	Velocity in Miles per Hour.									
	10	15	20	25	80	85	40	45	50	60
1	.006	.013	.028	.036	.051	.070	.091	.115	.143	.205
2	.011	.026	.046	.071	.103	.140	.182	.231	.285	.410
8	.017	.038	.068	.107	.154	.209	.274	.846	.428	.616
4	.023	.051	.091	.142	.205	.279	.365	.462	.570	.821
5	.029	.064	.114	.178	.257	.849	.456	.577	.712	
6	.034	.077	.187	.214	.308	.419	.547	.693	.855	
7	.040	.090	.160	.249	.859	.489	.638	.808	.997	
8	.046	.103	.182	.285	.410	.559	.730	.923		
9	.051	.115	.205	.321	.462	.628	.821			
10	.057	.128	.228	.356	.513	.698	.912			
12	.068	.153	.274	.428	.616	.838				
14	.080	.180	.319	.499	.718	.978				
16	.091	.205	.365	.570	.821					
18	.103	.231	.410	.641	.923					
20	.114	.257	.456	.712						
25	.143	.321	.570	.891						
30	.171	.385	.684							
85	.200	.449	.798							
40	.228	.513	.912							
50	.285	.641								

TABLE XIII.—COEFFICIENTS FOR REDUCING INCLINED STADIA MEASUREMENTS TO THE HORIZONTAL.

Inclination.	0′	10′	20′	30′	40′	50′
0°	1.00000	.99999	.99997	.99992	.99986	.99979
1	.99970	.99959	.99946	.99931	.99915	.99898
2	.99878	.99857	.99834	.99810	.99784	.99756
3	.99726	.99695	.99662	.99627	.99591	.99553
4	.99513	.99472	.99429	.99384	.99338	.99290
5	.99240	.99189	.99136	.99081	.99025	.98967
6	.98907	.98846	.98783	.98718	.98652	.98584
7	.98515	.98444	.98371	.98296	.98220	.98142
8	.98063	.97982	.97899	.97815	.97729	.97642
9	.97553	.97462	.97370	.97276	.97180	.97083
10	.96985	.96884	.96782	.96679	.96574	.96467
11	.96359	.96249	.96138	.96025	.95911	.95795
12	.95677	.95558	.95438	.95315	.95192	.95066
13	.94940	.94881	.94682	.94550	.94417	.94283
14	.94147	.94010	.93871	.93731	.93589	.93446
15	.93301	.93155	.93007	.92858	.92708	.92556
16	.92402	.92247	.92091	.91933	.91774	.91614
17	.91452	.91288	.91124	.90957	90790	.90621
18	.90451	.90279	.90106	.89932	.89756	.89579
19	.89400	.89221	.89040	.88857	.88673	.88488
20	.88302	.88114	.87926	.87735	.87544	.87351
21	.87157	.86962	.86765	.86568	.86369	.86168
22	.85967	.85764	.85560	.85355	.85149	.84941
23	.84783	.84523	.84312	.84100	.83886	.83672
24	.83456	.83240	.83022	.82803	.82583	.82361
25	.82139	.81916	.81691	.81466	.81239	.81011
26	.80783	.80553	.80322	.80091	.79858	.79624
27	.79389	.79153	.78916	.78679	.78440	.78200
28	.77959	.77718	.77475	.77232	.76987	.76742
29	.76496	.76249	.76001	.75752	.75502	.75251
30	.75000	.74747	.74494	.74240	.73985	.73730
31	.73473	.73216	.72958	.72699	.72440	.72179
32	.71918	.71656	.71394	.71131	.70867	.70602
33	.70336	.70070	.69804	.69536	.69268	.68999
34	.68730	.68460	.68189	.67918	.67646	.67374
35	.67101	.66827	.66553	.66278	.66003	.65727
36	.65450	.65174	.64896	.64618	.64340	.64061
37	.63781	.63502	.63221	.62941	.62659	.62378
38	.62096	.61813	.61530	.61247	.60963	.60679
39	.60395	.60110	.59825	.59540	.59254	.58968
40	.58682	.58395	.58108	.57821	.57534	.57246

TABLE XIV.—COEFFICIENTS FOR REDUCING GRADIENTER MEASUREMENTS TO THE HORIZONTAL.

Inclination.	0′	10′	20′	30′	40′	50′
0°	100.00	99.99	99.99	99.98	99.97	99.96
1	99.95	99.94	99.92	99.91	99.89	99.87
2	99.84	99.82	99.79	99.77	99.74	99.71
3	99.67	99.64	99.60	99.57	99.53	99.49
4	99.45	99.40	99.35	99.31	99.26	99.21
5	99.15	99.10	99.04	98.98	98.93	98.87
6	98.80	98.74	98.67	98.61	98.54	98.47
7	98.39	98.32	98.24	98.17	98.09	98.01
8	97.93	97.84	97.76	97.66	97.58	97.49
9	97.40	97.31	97.21	97.11	97.02	96.91
10	96.81	96.71	96.60	96.50	96.39	96.28
11	96.17	96.06	95.94	95.83	95.71	95.59
12	95.47	95.35	95.23	95.10	94.97	94.85
13	94.72	94.59	94.46	94.32	94.18	94.05
14	93.91	93.78	93.63	93.49	93.34	93.20
15	93.05	92.90	92.75	92.60	92.45	92.29
16	92.14	91.98	91.82	91.66	91.50	91.34
17	91.18	91.01	90.84	90.67	90.50	90.33
18	90.16	89.98	89.81	89.63	89.45	89.27
19	89.09	88.91	88.73	88.54	88.36	88.17
20	87.98	87.79	87.60	87.41	87.21	87.01
21	86.82	86.63	86.43	86.23	86.02	85.82
22	85.62	85.42	85.21	85.00	84.79	84.58
23	84.38	84.16	83.95	83.73	83.52	83.30
24	83.08	82.87	82.65	82.43	82.20	81.98
25	81.76	81.53	81.31	81.08	80.85	80.62
26	80.39	80.16	79.93	79.69	79.45	79.22
27	78.99	78.75	78.51	78.27	78.03	77.79
28	77.54	77.30	77.06	76.81	76.57	76.32
29	76.07	75.82	75.57	75.32	75.07	74.82
30	74.57	74.31	74.06	73.80	73.55	73.29

TABLE XV.—OFFSETS FOR TRANSITION CURVES.

$AK = t = \frac{S}{6} \sin \frac{O}{2}$. Fig. 180.

Deg. of Offset Curve.	Rad of Offset Curve	Lengths of Transition Curves.								
		100	150	200	240	250	300	320	360	400
0° 10′	34377	.01	.03	.05	.07	.08	.11	.12	.16	.19
20	17189	.02	.05	.10	.14	.15	.22	.25	.31	.39
30	11459	.04	.08	.15	.21	.22	.33	.37	.47	.58
40	8595 4	.05	.11	.19	.28	.30	.44	.50	.63	.78
50	6875.5	.06	.14	.24	.35	.38	.55	.62	.79	.97
1 00	5729.6	.07	.16	.29	.42	.46	.65	.74	.94	1.16
10	4911 1	.08	.19	.34	.49	.53	.76	.87	1.10	1.36
20	4297.2	.10	.22	.39	.56	.61	.87	.99	1.26	1.55
30	3819.7	.11	.25	.44	.63	.68	.98	1.12	1.41	1.75
40	3437.8	.12	.27	.48	.70	.76	1.09	1.24	1.57	1.94
50	3135.2	.13	.30	.53	.77	.83	1.20	1.37	1.73	2.13
2 00	2864.8	.15	.83	.58	.84	.91	1.31	1.49	1.88	2.33
10	2644.4	.16	.35	.63	.91	.98	1.42	1.61	2.04	2.52
20	2455.5	.17	.38	.68	.98	1.06	1.53	1.73	2.20	2.71
30	2291.8	.18	.41	.73	1.05	1.14	1.64	1.86	2.36	2.91
40	2148.6	.19	.44	.78	1.12	1.21	1.74	1.99	2.51	3.10
50	2022.2	.21	.46	.82	1.19	1.29	1.85	2.11	2 67	3.30
3 00	1909.9	.22	.49	.87	1.26	1.36	1.96	2.23	2.88	3.49
10	1809.3	.23	.52	.92	1.33	1.44	2.07	2.36	2.98	3.68
20	1718.9	.24	.55	.97	1.40	1.51	2.18	2.48	3.14	3.88
30	1637.0	.25	.57	1.02	1.47	1.59	2.29	2.61	3.30	4.07
40	1562.6	.27	.60	1.07	1.54	1.67	2.40	2.73	3.45	4.26
50	1494.7	.28	.63	1.11	1.61	1.74	2.51	2.85	3.61	4.46
4 00	1432.4	.29	.65	1.16	1.68	1.82	2.62	2.98	3.77	4.65
10	1375.1	.30	.68	1.21	1.74	1.89	2.73	3.10	3.92	4.84
20	1322.2	.32	.71	1.26	1.81	1.97	2.83	3.23	4.08	5.04
30	1273.2	.33	.74	1.31	1.88	2.04	2.94	3.35	4.24	5.23
40	1227.8	.34	.76	1.36	1.95	2.12	3.05	3.47	4.39	5.42
50	1185.4	.35	.79	1.41	2.02	2.20	3.16	3.60	4.55	5.62
5 00	1145.9	.36	.82	1.45	2.09	2.27	3.27	3.72	4.71	5.81
10	1109.0	.38	.85	1.50	2.16	2.35	3.38	3.84	4.86	6.00
20	1074.3	.39	.87	1.55	2.23	2.42	3.49	3.97	5.02	6.20
30	1041.7	.40	.90	1.60	2.30	2.49	3.60	4.09	5.18	6.39
40	1011.1	.41	.93	1.65	2.37	2.57	3.71	4.22	5.33	6.58
50	982.2	.42	.95	1.70	2.44	2.65	3.81	4.34	5 49	6.78
6 00	954.9	.44	.98	1.74	2.51	2.73	3.92	4.46	5.65	6.97
10	929.1	.45	1.01	1 79	2.58	2.80	4.03	4.59	5.80	7.16
20	904.7	.46	1.04	1.84	2.65	2.88	4.14	4.71	5.96	7.36
30	881.5	.47	1.06	1.89	2.72	2.95	4.25	4.83	6.12	7.55
40	859.4	.48	1.09	1.94	2.79	3.03	4.36	4.96	6.27	7.74
50	838.5	.50	1.12	1.99	2.86	3.10	4.47	5.08	6.43	7.93
7 00	818.5	.51	1.14	2.03	2.93	3.18	4.58	5.20	6.58	8.12
10	799.5	.52	1.17	2.08	3.00	3.25	4.68	5.33	6.74	8.32
20	781.3	.53	1.20	2.13	3.07	3.33	4.79	5.45	6.90	8.51
30	763.9	.55	1.23	2.18	3.14	3.41	4.90	5.57	7.05	8.70
40	747.3	.56	1.25	2.23	3.21	3.48	5.01	5.70	7.21	8.89
50	731.4	.57	1.28	2.28	3.28	3.56	5.12	5.82	7.36	9.09
8 00	716.2	.58	1.31	2.33	3.85	3.63	5.23	5.95	7.52	9 28
10	701.6	.59	1.34	2.37	3.42	3.71	5.33	6.07	7.68	9.47
20	687.5	.61	1.86	2.42	3 49	3.78	5.44	6.19	7.83	9.66
30	674.1	.62	1.39	2.47	3.56	3.86	5.55	6.31	7.99	9.85
40	661.1	.63	1.42	2.52	3.63	3.93	5.66	6.44	8.14	10.05
50	648.6	.64	1.44	2.57	3.69	4.01	5.77	6.56	8.30	10.24
9 00	636.6	.65	1.47	2.62	3.76	4.08	5.88	6.68	8.45	10.43
10	625.0	.67	1.50	2.66	3 83	4.16	5.99	6.81	8.61	10.62
20	613.9	.68	1.53	2.71	3.90	4.23	6.09	6.93	8.76	10.81
30	603.1	.69	1.55	2.76	3.97	4.31	6.20	7.05	8.92	11.00
40	592.7	.70	1.58	2.81	4.04	4.39	6.31	7.18	9.08	11.19
50	582.7	.71	1.61	2.86	4.11	4.46	6.42	7.30	9.23	11.39
10 00	573.0	.73	1.64	2.91	4.18	4.54	6.53	7.42	9.39	11.58

Deg. of Offset Curve.	Rad. of Offset Curve.	Lengths of Transition Curves.								
		100	150	200	240	250	300	320	360	400
10° 10′	563.6	.74	1.66	2.95	4.25	4.61	6.63	7.55	9.54	11.77
20	554.5	.75	1.69	3.00	4.32	4.69	6.74	7.67	9.70	11.96
30	545.7	.76	1.72	3.05	4.39	4.76	6.85	7.79	9.85	12.15
40	537.1	.78	1.74	3.10	4.46	4.84	6.96	7.91	10.0	12.34
50	528.9	.79	1.77	3.15	4.53	4.91	7.07	8.03	10.16	12.53
11 00	520.9	.80	1.80	3.19	4.60	4.99	7.17	8.16	10.3	12.7
10	513.1	.81	1.83	3.24	4.67	5.06	7.28	8.28	10.47	12.91
20	505.6	.82	1.85	3.29	4.74	5.14	7.39	8.40	10.62	13.10
30	498.2	.84	1.88	3.34	4.81	5.21	7.50	8.53	10.78	13.29
40	491.1	.85	1.91	3.39	4.87	5.29	7.61	8.65	10.93	13.48
50	484.2	.86	1.93	3.44	4.94	5.36	7.71	8.77	11.09	13.67
12 00	477.5	.87	1.96	3.48	5.01	5.44	7.82	8.89	11.24	13.86
10	470.9	.88	1.99	3.53	5.08	5.51	7.93	9.02	11.40	14.05
20	464.6	.90	2.02	3.58	5.15	5.59	8.04	9.14	11.55	14.24
30	458.4	.91	2.04	3.63	5.22	5.66	8.14	9.26	11.71	14.43
40	452.3	.92	2.07	3.68	5.29	5.74	8.25	9.38	11.86	14.62
50	446.5	.93	2.10	3.73	5.36	5.81	8.36	9.51	12.01	14.81
13 00	440.7	.94	2.12	3.77	5.43	5.89	8.47	9.63	12.17	15.00
10	435.2	.96	2.15	3.82	5.50	5.96	8.58	9.75	12.32	15.19
20	429.7	.97	2.18	3.87	5.57	6.04	8.68	9.87	12.47	15.37
30	424.4	.98	2.21	3.92	5.64	6.11	8.79	9.99	12.63	15.56
40	419.2	.99	2.23	3.97	5.71	6.19	8.90	10.12	12.78	15.75
50	414.2	1.01	2.26	4.01	5.77	6.26	9.00	10.23	12.94	15.94
14 00	409.3	1.02	2.29	4.06	5.84	6.34	9.11	10.36	13.09	16.13
10	404.4	1.03	2.31	4.11	5.91	6.41	9.22	10.48	13.24	16.32
20	399.7	1.04	2.34	4.16	5.98	6.49	9.33	10.60	13.40	16.50
30	395.1	1.05	2.37	4.21	6.05	6.56	9.43	10.72	13.55	16.69
40	390.7	1.07	2.40	4.25	6.12	6.64	9.54	10.85	13.70	16.88
50	385.3	1.08	2.42	4.30	6.19	6.71	9.65	10.97	13.85	17.07
15 00	382.0	1.09	2.45	4.35	6.26	6.79	9.75	11.09	14.01	17.25
10	377.8	1.10	2.48	4.40	6.33	6.86	9.86	11.21	14.16	17.44
20	373.7	1.11	2.50	4.45	6.40	6.94	9.97	11.33	14.31	17.63
30	369.7	1.13	2.53	4.50	6.46	7.01	10.08	11.45	14.46	17.82
40	365.7	1.14	2.56	4.54	6.53	7.09	10.18	11.57	14.62	18.00
50	361.9	1.15	2.59	4.59	6.60	7.16	10.29	11.69	14.77	18.19
16 00	358.1	1.16	2.61	4.64	6.67	7.24	10.40	11.82	14.92	18.38
10	354.4	1.17	2.64	4.69	6.74	7.31	10.50	11.94	15.07	18.56
20	350.8	1.19	2.67	4.74	6.81	7.38	10.61	12.06	15.23	18.75
30	347.2	1.20	2.69	4.78	6.88	7.48	10.72	12.18	15.38	18.93
40	343.8	1.21	2.72	4.83	6.95	7.53	10.82	12.30	15.53	19.12
50	340.4	1.22	2.75	4.88	7.01	7.61	10.93	12.42	15.68	19.31
17 00	337.0	1.24	2.78	4.93	7.08	7.68	11.03	12.54	15.83	19.49
10	333.8	1.25	2.80	4.98	7.15	7.76	11.14	12.66	15.98	19.68
20	330.6	1.26	2.83	5.02	7.22	7.83	11.25	12.78	16.14	19.86
30	327.4	1.27	2.86	5.07	7.29	7.91	11.35	12.90	16.29	20.05
40	324.3	1.28	2.88	5.12	7.36	7.98	11.46	13.02	16.44	20.23
50	321.3	1.30	2.91	5.17	7.43	8.05	11.57	13.14	16.59	20.42
18 00	318.3	1.31	2.94	5.21	7.50	8.13	11.67	13.26	16.74	20.60
10	315.4	1.32	2.97	5.26	7.56	8.20	11.78	13.38	16.89	20.79
20	312.5	1.33	2.99	5.31	7.63	8.28	11.88	13.50	17.04	20.97
30	309.7	1.34	3.02	5.36	7.70	8.35	11.99	13.62	17.19	21.15
40	306.9	1.36	3.05	5.41	7.77	8.43	12.10	18.74	17.34	21.34
50	304.2	1.37	3.07	5.45	7.84	8.50	12.20	13.86	17.49	21.52
19 00	301.6	1.38	3.10	5.50	7.91	8.57	12.31	13.98	17.64	21.70
10	298.9	1.39	3.13	5.55	7.97	8.65	12.41	14.10	17.79	21.89
20	296.4	1.40	3.16	5.60	8.04	8.72	12.52	14.22	17.94	22.07
30	293.8	1.42	3.18	5.65	8.11	8.80	12.62	14.34	18.09	22.25
40	291.3	1.43	3.21	5.69	8.18	8.87	12.73	14.46	18.24	22.44
50	288.9	1.44	3.24	5.74	8.25	8.94	12.84	14.58	18.39	22.62
20 00	286.5	1.45	3.26	5.79	8.32	9.02	12.94	14.70	18.54	22

Table gives $d = \frac{S}{2} \text{ vers } \frac{O}{4}$. Then $AO = \frac{S}{2} - d$.

Degree of Offset Curve.	Lengths of Transition Curves.								
	100	**150**	**200**	**240**	**250**	**300**	**320**	**360**	**400**
0° 10′									
20								.001	.001
30						.001	.001	.001	.002
40				.001	.001	.001	.002	.002	.003
50			.001	.001	.001	.002	.003	.004	.005
1 00			.001	.002	.002	.003	.004	.006	.008
10		.001	.001	.002	.003	.005	.005	.008	.01
20		.001	.002	.003	.003	.006	.007	.01	.01
30		.001	.002	.004	.004	.007	.009	.01	.02
40		.001	.003	.005	.005	.009	.01	.02	.02
50		.001	.003	.006	.006	.01	.01	.02	.03
2 00		.002	.004	.007	.007	.01	.02	.02	.03
10	.001	.002	.005	.008	.009	.02	.02	.03	.04
20	.001	.002	.005	.009	.01	.02	.02	.03	.04
30	.001	.002	.006	.01	.01	.02	.02	.03	.05
40	.001	.003	.007	.01	.01	.02	.03	.04	.05
50	.001	.003	.008	.01	.01	.03	.03	.04	.06
3 00	.001	.004	.009	.01	.02	.03	.04	.05	.07
10	.001	.004	.01	.02	.02	.03	.04	.06	.08
20	.001	.004	.01	.02	.02	.04	.04	.06	.08
30	.001	.005	.01	.02	.02	.04	.05	.07	.09
40	.002	.005	.01	.02	.03	.04	.05	.07	.10
50	.002	.006	.01	.02	.03	.05	.06	.08	.11
4 00	.002	.006	.02	.03	.03	.05	.06	.09	.12
10	.002	.007	.02	.03	.03	.06	.07	.10	.13
20	.002	.008	.02	.03	.03	.06	.07	.10	.14
30	.002	.008	.02	.03	.04	.07	.08	.11	.15
40	.003	.009	.02	.04	.04	.07	.08	.12	.17
50	.003	.009	.02	.04	.04	.08	.09	.13	.18
5 00	.003	.01	.02	.04	.05	.08	.10	.14	.19
10	.003	.01	.03	.04	.05	.09	.10	.15	.20
20	.003	.01	.03	.05	.05	.09	.11	.16	.22
30	.004	.01	.03	.05	.06	.10	.12	.17	.23
40	.004	.01	.03	.05	.06	.10	.13	.18	.24
50	.004	.01	.03	.06	.06	.11	.13	.19	.26
6 00	.004	.01	.03	.06	.07	.12	.14	.20	.27
10	.005	.02	.04	.06	.07	.12	.15	.21	.29
20	.005	.02	.04	.07	.07	.13	.16	.22	.31
30	.005	.02	.04	.07	.08	.14	.16	.23	.32
40	.005	.02	.04	.07	.08	.14	.17	.25	.34
50	.006	.02	.04	.08	.09	.15	.18	.26	.36
7 00	.006	.02	.05	.08	.09	.16	.19	.27	.37
10	.006	.02	.05	.08	.09	.16	.20	.28	.39
20	.006	.02	.05	.09	.10	.17	.21	.30	.41
30	.007	.02	.05	.09	.10	.18	.22	.31	.43
40	.007	.02	.06	.10	.11	.19	.23	.33	.45
50	.007	.02	.06	.10	.11	.20	.24	.34	.47
8 00	.008	.03	.06	.11	.12	.21	.25	.36	.49
10	.008	.03	.06	.11	.12	.21	.26	.37	.51
20	.008	.03	.07	.11	.13	.22	.27	.39	.53
30	.009	.03	.07	.12	.13	.23	.28	.40	.55
40	.009	.03	.07	.12	.14	.24	.29	.42	.57
50	.009	.03	.07	.13	.15	.25	.30	.43	.59
9 00	.01	.03	.08	.13	.15	.26	.32	.45	.62
10	.01	.03	.08	.14	.16	.27	.33	.47	.64
20	.01	.03	.08	.14	.16	.28	.34	.48	.66
30	.01	.04	.09	.15	.17	.29	.35	.50	.69
40	.01	.04	.09	.15	.17	.30	.36	.52	.71
50	.01	.04	.09	.16	.18	.31	.38	.54	.74
00	.01	.04	.10	.16	.19	.32	.39	.55	.76

Degree of Offset Curve.	Lengths of Transition Curves.								
	100	150	200	240	250	300	320	360	400
10° 10′	.01	.04	.10	.17	.19	.33	.40	.57	.79
20	.01	.04	.10	.18	.20	.34	.42	.59	.81
30	.01	.04	.10	.18	.20	.35	.43	.61	.84
40	.01	.05	.11	.19	.21	.37	.44	.63	.87
50	.01	.05	.11	.19	.22	.38	.46	.65	.89
11 00	.01	.05	.12	.20	.23	.39	.47	.67	.92
10	.01	.05	.12	.21	.23	.40	.49	.69	.95
20	.02	.05	.12	.21	.24	.41	.50	.71	.98
30	.02	.05	.13	.22	.25	.42	.52	.73	1.01
40	.02	.05	.13	.22	.25	.44	.53	.76	1.04
50	.02	.06	.13	.23	.26	.45	.55	.78	1.07
12 00	.02	.06	.14	.24	.27	.46	.56	.80	1.10
10	.02	.06	.14	.24	.27	.48	.58	.82	1.13
20	.02	.06	.14	.25	.28	.49	.59	.84	1.16
30	.02	.06	.15	.26	.29	.50	.61	.87	1.19
40	.02	.06	.15	.26	.30	.52	.63	.89	1.22
50	.02	.07	.16	.27	.31	.53	.64	.91	1.25
13 00	.02	.07	.16	.28	.32	.54	.66	.94	1.29
10	.02	.07	.16	.29	.32	.55	.68	.96	1.32
20	.02	.07	.17	.29	.33	.57	.69	.99	1.35
30	.02	.07	.17	.30	.34	.59	.71	1.01	1.39
40	.02	.07	.18	.31	.35	.60	.73	1.04	1.42
50	.02	.08	.18	.31	.36	.61	.75	1.06	1.46
14 00	.02	.08	.19	.32	.36	.62	.76	1.09	1.49
10	.02	.08	.19	.33	.37	.64	.78	1.11	1.53
20	.02	.08	.20	.34	.38	.66	.80	1.14	1.56
30	.03	.08	.20	.35	.39	.68	.82	1.17	1.60
40	.03	.09	.20	.35	.40	.69	.84	1.19	1.64
50	.03	.09	.21	.36	.41	.71	.86	1.22	1.67
15 00	.03	.09	.21	.37	.42	.72	.88	1.25	1.71
10	.03	.09	.22	38	.43	.74	.90	1.28	1.75
20	.03	.09	.22	.39	.44	.75	.92	1.30	1.79
30	.03	.10	.23	.39	.45	.77	.94	1.33	1.83
40	.03	.10	.23	.40	.46	.79	.96	1.36	1.87
50	.03	.10	.24	.41	.47	.80	.98	1.39	1.91
16 00	.03	.10	.24	.42	.48	.82	1.00	1.42	1.95
10	.03	.10	.25	.43	.49	.84	1.02	1.45	1.99
20	.03	.11	.25	.44	.50	.86	1.04	1.48	2.03
30	.03	.11	.26	.45	.51	.87	1.06	1.51	2.07
40	.03	.11	.26	.46	.52	.89	1.08	1.54	2.11
50	.03	.11	.27	.47	.53	.91	1.10	1.57	2.15
17 00	.03	.12	.27	.48	.54	.93	1.13	1.60	2.20
10	.04	.12	.28	.48	.55	.95	1.15	1.63	2.24
20	.04	.12	.29	.49	.56	.96	1.17	1.66	2.28
30	.04	.12	.29	.50	.57	.98	1.19	1.70	2.33
40	.04	.13	.30	.51	.58	1.00	1.22	1.73	2.37
50	.04	.13	.30	.52	.59	1.02	1.24	1.76	2.42
18 00	.04	.13	.31	.53	.60	1.04	1.26	1.80	2.46
10	.04	.13	.31	.54	.61	1.06	1.29	1.83	2.51
20	.04	.13	.32	.55	.62	1.08	1.31	1.86	2.55
30	.04	.14	.33	.56	.64	1.10	1.33	1.90	2.60
40	.04	.14	.33	.57	.65	1.12	1.36	1.93	2.65
50	.04	.14	.34	.58	.66	1.14	1.38	1.96	2.70
19 00	.04	.14	.34	.59	.67	1.16	1.41	2.00	2.74
10	.04	.15	.35	.60	.68	1.18	1.43	2.04	2.79
20	.04	.15	.36	.61	.69	1.20	1.46	2.07	2.84
30	.05	.15	.36	.63	.71	1.22	1.48	2.11	2.89
40	.05	.16	.37	.64	.72	1.24	1.51	2.14	2.94
50	.05	.16	.37	.65	.73	1.26	1.53	2.18	2.99
20 00	.05	.16	.38	.66	.74	1.28	1.56	2.22	3.04

TABLE XVII.—DEFLECTION ANGLES FOR TRANSITION CURVES, TO TENTHS OF A MINUTE, FOR CHORDS OF 1 TO 5 CHAINS, IN TERMS OF ANGLE FOR CHORD OF 1 CHAIN.

1	2	3	4	5	1	2	3	4	5
0′.1	0′.4	0′.9	1′.6	2′.5	6′.1	24′.4	54′.9	1° 38′	2° 32′
0 .2	0 .8	1 .8	3 .2	5 .0	6 .2	24 .8	55 .8	39	35
0 .3	1 .2	2 .7	4 .8	7 .5	6 .3	25 .2	56 .7	41	37
0 .4	1 .6	3 .6	6 .4	10 .0	6 .4	25 .6	57 .6	42	40
0 .5	2 .0	4 .5	8 .0	12 .5	6 .5	26 .0	58 .5	44	42
0 .6	2 .4	5 .4	9 .6	15 .0	6 .6	26 .4	59 .4	46	45
0 .7	2 .8	6 .3	11 .2	17 .5	6 .7	26 .8	1° 0	47	47
0 .8	3 .2	7 .2	12 .8	20 .0	6 .8	27 .2	1	49	50
0 .9	3 .6	8 .1	14 .4	22 .5	6 .9	27 .6	2	50	52
1 .0	4 .0	9 .0	16 .0	25 .0	7 .0	28 .0	3	52	55
1 .1	4 .4	9 .9	17 .6	27 .5	7 .1	28 .4	4	54	57
1 .2	4 .8	10 .8	19 .2	30 .0	7 .2	28 .8	5	55	3 0
1 .3	5 .2	11 .7	20 .8	32 .5	7 .3	29 .2	6	57	2
1 .4	5 .6	12 .6	22 .4	35 .0	7 .4	29 .6	7	58	5
1 .5	6 .0	13 .5	24 .0	37 .5	7 .5	30 .0	8	2 0	7
1 .6	6 .4	14 .4	25 .6	40 .0	7 .6	30 .4	8	2	10
1 .7	6 .8	15 .3	27 .2	42 .5	7 .7	30 .8	9	3	12
1 .8	7 .2	16 .2	28 .8	45 .0	7 .8	31 .2	10	5	15
1 .9	7 .6	17 .1	30 .4	47 .5	7 .9	31 .6	11	6	17
2 .0	8 .0	18 .0	32 .0	50 .0	8 .0	32 .0	12	8	20
2 .1	8 .4	18 .9	33 .6	52 .5	8 .1	32 .4	13	10	22
2 .2	8 .8	19 .8	35 .2	55 .0	8 .2	32 .8	14	11	25
2 .3	9 .2	20 .7	36 .8	57 .5	8 .3	33 .2	15	13	27
2 .4	9 .6	21 .6	38 .4	1° 0′	8 .4	33 .6	16	14	30
2 .5	10 .0	22 .5	40 .0	2	8 .5	34 .0	17	16	32
2 .6	10 .4	23 .4	41 .6	5	8 .6	34 .4	17	18	35
2 .7	10 .8	24 .3	43 .2	7	8 .7	34 .8	18	19	37
2 .8	11 .2	25 .2	44 .8	10	8 .8	35 .2	19	21	40
2 .9	11 .6	26 .1	46 .4	12	8 .9	35 .6	20	22	42
3 .0	12 .0	27 .0	48 .0	15	9 .0	36 .0	21	24	45
3 .1	12 .4	27 .9	49 .6	17	9 .1	36 .4	22	26	47
3 .2	12 .8	28 .8	51 .2	20	9 .2	36 .8	23	27	50
3 .3	13 .2	29 .7	52 .8	22	9 .3	37 .2	24	29	52
3 .4	13 .6	30 .6	54 .4	25	9 .4	37 .6	25	30	55
3 .5	14 .0	31 .5	56 .0	27	9 .5	38 .0	26	32	57
3 .6	14 .4	32 .4	57 .6	30	9 .6	38 .4	26	34	4 0
3 .7	14 .8	33 .3	59 .2	32	9 .7	38 .8	27	35	2
3 .8	15 .2	34 .2	1° 1′	35	9 .8	39 .2	28	37	5
3 .9	15 .6	35 .1	2	37	9 .9	39 .6	29	38	7
4 .0	16 .0	36 .0	4	40	10 .0	40 .0	30	40	10
4 .1	16 .4	36 .9	6	42	10 .1	40 .4	31	42	12
4 .2	16 .8	37 .8	7	45	10 .2	40 .8	32	43	15
4 .3	17 .2	38 .7	9	47	10 .3	41 .2	33	45	17
4 .4	17 .6	39 .6	10	50	10 .4	41 .6	34	46	20
4 .5	18 .0	40 .5	12	52	10 .5	42 .0	35	48	22
4 .6	18 .4	41 .4	14	55	10 .6	42 .4	35	50	25
4 .7	18 .8	42 .3	15	57	10 .7	42 .8	36	51	27
4 .8	19 .2	43 .2	17	2 0	10 .8	43 .2	37	53	30
4 .9	19 .6	44 .1	18	2	10 .9	43 .6	38	54	32
5 .0	20 .0	45 .0	20	5	11 .0	44 .0	39	56	35
5 .1	20 .4	45 .9	22	7	11 .1	44 .4	40	58	37
5 .2	20 .8	46 .8	23	10	11 .2	44 .8	41	59	40
5 .3	21 .2	47 .7	25	12	11 .3	45 .2	42	3 1	42
5 .4	21 .6	48 .6	26	15	11 .4	45 .6	43	2	45
5 .5	22 .0	49 .5	28	17	11 .5	46 .0	44	4	47
5 .6	22 .4	50 .4	30	20	11 .6	46 .4	44	6	50
5 .7	22 .8	51 .3	31	22	11 .7	46 .8	45	7	52
5 .8	23 .2	52 .2	33	25	11 .8	47 .2	46	9	55
5 .9	23 .6	53 .1	34	27	11 .9	47 .6	47	10	57
6 .0	24 .0	54 .0	36	30	12 .0	48 .0	48	12	5 0

BASE = 8. SIDE SLOPES 2 TO 1.

D	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0		3	6	10	13	17	20	24	28	33
1	37	42	46	51	56	61	66	72	77	83
2	89	95	101	107	114	120	127	134	141	148
3	156	163	171	178	186	194	203	211	220	228
4	237	246	255	264	274	283	293	303	313	323
5	333	344	354	365	376	387	398	410	421	433
6	444	456	468	481	493	506	518	531	544	557
7	570	584	597	611	625	639	653	667	682	696
8	711	726	741	756	772	787	803	818	834	850
9	867	883	900	916	933	950	967	984	1002	1019
10	1037	1055	1073	1091	1109	1128	1146	1165	1184	1203
11	1222	1242	1261	1281	1300	1320	1340	1361	1381	1402
12	1422	1443	1464	1485	1506	1528	1549	1571	1593	1615
13	1637	1659	1682	1704	1727	1750	1773	1796	1820	1843
14	1867	1890	1914	1938	1963	1987	2012	2036	2061	2086
15	2111	2136	2162	2187	2213	2239	2265	2291	2317	2344
16	2370	2397	2424	2451	2478	2506	2533	2561	2588	2616
17	2644	2673	2701	2730	2758	2787	2816	2845	2874	2904
18	2933	2963	2993	3023	3053	3083	3114	3144	3175	3206
19	3237	3268	3300	3331	3363	3394	3426	3458	3491	3523
20	3556	3588	3621	3654	3687	3720	3754	3787	3821	3855
21	3889	3923	3957	3992	4026	4061	4096	4131	4166	4202
22	4237	4273	4308	4344	4380	4417	4453	4490	4526	4563
23	4600	4637	4674	4712	4749	4787	4825	4863	4901	4939
24	4978	5016	5055	5094	5133	5172	5212	5251	5291	5330
25	5370	5410	5451	5491	5532	5572	5613	5654	5695	5736
26	5778	5819	5861	5903	5945	5987	6029	6072	6114	6157
27	6200	6243	6286	6330	6373	6417	6460	6504	6548	6593
28	6637	6682	6726	6771	6816	6861	6906	6952	6997	7043
29	7089	7135	7181	7227	7274	7320	7367	7414	7461	7508

BASE = 8. SIDE SLOPES 3 TO 1.

0		3	6	10	14	18	22	26	31	36
1	41	46	52	57	63	69	76	82	89	96
2	104	111	119	127	135	144	152	161	170	179
3	189	199	209	219	229	240	251	262	273	285
4	296	308	320	333	345	358	371	385	398	412
5	426	440	455	469	484	499	514	530	546	562
6	578	594	611	628	645	662	680	697	715	733
7	752	770	789	808	828	847	867	887	907	928
8	948	969	990	1011	1033	1055	1077	1099	1121	1144
9	1167	1190	1213	1237	1260	1284	1308	1333	1357	1382
10	1407	1433	1458	1484	1510	1536	1563	1589	1616	1643
11	1670	1698	1726	1754	1782	1810	1839	1868	1897	1926
12	1956	1985	2015	2045	2076	2106	2137	2168	2200	2231
13	2263	2295	2327	2360	2392	2425	2458	2491	2525	2559
14	2593	2627	2661	2696	2731	2766	2801	2837	2872	2908
15	2944	2981	3017	3054	3091	3129	3166	3204	3242	3280
16	3319	3357	3396	3435	3474	3514	3554	3594	3634	3674
17	3715	3756	3797	3838	3880	3921	3963	4005	4048	4090
18	4133	4176	4220	4263	4307	4351	4395	4440	4484	4529
19	4574	4619	4665	4711	4757	4803	4849	4896	4943	4990
20	5037	5085	5132	5180	5228	5277	5325	5374	5423	5473
21	5522	5572	5622	5672	5723	5773	5824	5875	5926	5978
22	6030	6082	6134	6186	6239	6292	6345	6398	6452	6505
23	6559	6613	6668	6722	6777	6832	6888	6943	6999	7055
24	7111	7168	7224	7281	7338	7395	7453	7511	7569	7627
25	7685	7744	7803	7862	7921	7981	8040	8100	8160	8221
26	8281	8342	8403	8465	8526	8588	8650	8712	8775	8837
27	8900	8963	9026	9090	9154	9218	9282	9346	9411	9476
28	9541	9606	9672	9737	9802	9869	9936	10002	10069	10136
29	10204	10271	10339	10407	10475	10544	10612	10681	10750	10819

BASE = 14. SIDE SLOPES $\frac{3}{2}$ TO 1.

D	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0		5	11	16	22	27	33	39	45	51
1	57	64	70	77	83	90	97	104	111	119
2	126	133	141	149	156	164	172	181	189	197
3	206	214	223	232	241	250	259	268	277	287
4	296	306	316	326	336	346	356	366	377	387
5	398	409	420	431	442	453	465	476	488	499
6	511	523	535	547	559	572	584	597	609	622
7	635	648	661	675	688	701	715	729	742	756
8	770	785	799	813	828	842	857	872	887	902
9	917	932	947	963	978	994	1010	1026	1042	1058
10	1074	1090	1107	1123	1140	1157	1174	1191	1208	1225
11	1243	1260	1278	1295	1313	1331	1349	1367	1385	1404
12	1422	1441	1459	1478	1497	1516	1535	1555	1574	1593
13	1613	1633	1652	1672	1692	1713	1733	1753	1774	1794
14	1815	1836	1857	1878	1899	1920	1941	1963	1984	2006
15	2028	2050	2072	2094	2116	2138	2161	2183	2206	2229
16	2252	2275	2298	2321	2345	2368	2392	2415	2439	2463
17	2487	2511	2535	2560	2584	2609	2633	2658	2683	2708
18	2733	2759	2784	2809	2835	2861	2886	2912	2938	2965
19	2991	3017	3044	3070	3097	3124	3151	3178	3205	3232
20	3259	3287	3314	3342	3370	3398	3426	3454	3482	3510
21	3539	3567	3596	3625	3654	3683	3712	3741	3771	3800
22	3830	3859	3889	3919	3949	3979	4009	4040	4070	4101
23	4131	4162	4193	4224	4255	4287	4318	4349	4381	4413
24	4444	4476	4508	4541	4573	4605	4638	4670	4703	4736
25	4769	4802	4835	4868	4901	4935	4968	5002	5036	5070
26	5104	5138	5172	5206	5241	5275	5310	5345	5380	5415
27	5450	5485	5521	5556	5592	5627	5663	5699	5735	5771
28	5807	5844	5880	5917	5953	5990	6027	6064	6101	6139
29	6176	6213	6251	6289	6326	6364	6402	6441	6479	6517

BASE = 10. SIDE SLOPES 1 TO 1.

D	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0		4	8	11	15	19	24	28	32	36
1	41	45	50	54	59	64	69	74	79	84
2	89	94	99	105	110	116	121	127	133	139
3	144	150	156	163	169	175	181	188	194	201
4	207	214	221	228	235	242	249	256	263	270
5	278	285	293	300	308	316	324	331	339	347
6	356	364	372	380	389	397	406	414	423	432
7	441	450	459	468	477	486	495	505	514	524
8	533	543	553	563	572	582	592	603	613	623
9	633	644	654	665	675	686	697	708	719	730
10	741	752	763	774	786	797	809	820	832	844
11	856	867	879	891	904	916	928	940	953	965
12	978	990	1003	1016	1029	1042	1055	1068	1081	1094
13	1107	1121	1134	1148	1161	1175	1189	1203	1216	1230
14	1244	1259	1273	1287	1301	1316	1330	1345	1359	1374
15	1389	1404	1419	1434	1449	1464	1479	1494	1510	1525
16	1541	1556	1572	1588	1604	1619	1635	1651	1668	1684
17	1700	1716	1733	1749	1766	1782	1799	1816	1833	1850
18	1867	1884	1901	1918	1935	1953	1970	1988	2005	2023
19	2041	2059	2076	2094	2112	2131	2149	2167	2185	2204
20	2222	2241	2259	2278	2297	2316	2335	2354	2373	2392
21	2411	2430	2450	2469	2489	2508	2528	2548	2568	2587
22	2607	2627	2648	2668	2688	2708	2729	2749	2770	2790
23	2811	2832	2853	2874	2895	2916	2937	2958	2979	3001
24	3022	3044	3065	3087	3109	3131	3152	3174	3196	3219
25	3241	3263	3285	3308	3330	3353	3375	3398	3421	3444
26	3467	3490	3513	3536	3559	3582	3606	3629	3653	3676
27	3700	3724	3748	3771	3795	3819	3844	3868	3892	3916
28	3941	3965	3990	4014	4039	4064	4089	4114	4139	4164
29	4189	4214	4239	4265	4290	4316	4341	4367	4393	4419

BASE = 16. SIDE SLOPES $\frac{3}{2}$ TO 1.

D	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0		6	12	18	25	31	38	44	51	58
1	65	72	79	86	94	101	109	117	125	133
2	141	149	157	166	174	183	192	201	209	219
3	228	237	247	256	266	275	285	295	305	316
4	326	336	347	358	368	379	390	401	412	424
5	435	447	458	470	482	494	506	518	531	543
6	556	568	581	594	607	620	633	646	660	673
7	687	701	715	729	743	757	771	786	800	815
8	830	845	859	875	890	905	921	936	952	967
9	983	999	1015	1032	1048	1064	1081	1098	1114	1131
10	1148	1165	1182	1200	1217	1235	1252	1270	1288	1306
11	1324	1342	1361	1379	1398	1416	1435	1454	1473	1492
12	1511	1530	1550	1569	1589	1609	1629	1649	1669	1689
13	1709	1730	1750	1771	1792	1813	1883	1855	1876	1897
14	1919	1940	1962	1983	2005	2027	2049	2072	2094	2116
15	2139	2162	2184	2207	2230	2253	2276	2300	2323	2347
16	2370	2394	2418	2442	2466	2490	2515	2539	2564	2588
17	2613	2638	2663	2688	2713	2738	2764	2789	2815	2841
18	2867	2893	2919	2945	2971	2998	3024	3051	3078	3105
19	3131	3159	3186	3213	3241	3268	3296	3323	3351	3379
20	3407	3436	3464	3492	3521	3550	3578	3607	3636	3665
21	3694	3724	3753	3783	3812	3842	3872	3902	3932	3962
22	3993	4023	4054	4084	4115	4146	4177	4208	4239	4270
23	4302	4333	4365	4397	4429	4461	4493	4525	4557	4590
24	4622	4655	4688	4721	4753	4787	4820	4853	4887	4920
25	4954	4987	5021	5055	5089	5124	5158	5192	5227	5262
26	5296	5331	5366	5401	5436	5472	5507	5543	5578	5614
27	5650	5686	5722	5758	5795	5831	5868	5904	5941	5978
28	6015	6052	6089	6126	6164	6201	6239	6277	6315	6353
29	6391	6429	6467	6506	6544	6583	6622	6661	6699	6739

BASE = 18. SIDE SLOPES 1 TO 1.

0		7	13	20	27	34	41	48	56	63
1	70	78	85	93	101	108	116	124	132	140
2	148	156	165	173	181	190	198	207	216	224
3	283	242	251	260	269	279	288	297	307	316
4	326	336	345	355	365	375	385	395	405	416
5	426	436	447	457	468	479	489	500	511	522
6	533	544	556	567	578	590	601	613	625	636
7	648	660	672	684	696	708	721	733	745.	758
8	770	783	796	808	821	834	847	860	873	887
9	900	913	927	940	954	968	981	995	1009	1023
10	1037	1051	1065	1080	1094	1108	1123	1137	1152	1167
11	1181	1196	1211	1226	1241	1256	1272	1287	1302	1318
12	1333	1349	1365	1380	1396	1412	1428	1444	1460	1476
13	1493	1509	1525	1542	1558	1575	1592	1608	1625	1642
14	1659	1676	1693	1711	1728	1745	1763	1780	1798	1816
15	1833	1851	1869	1887	1905	1923	1941	1960	1978	1996
16	2015	2033	2052	2071	2089	2108	2127	2146	2165	2184
17	2204	2223	2242	2262	2281	2301	2321	2340	2360	2380
8	2400	2420	2440	2460	2481	2501	2521	2542	2562	2583
19	2604	2624	2645	2666	2687	2708	2729	2751	2772	2793
20	2815	2836	2858	2880	2901	2923	2945	2967	2989	3011
21	3033	3056	3078	3100	3123	3145	3168	3191	3213	3236
22	3259	3282	3305	3328	3352	3375	3398	3422	8445	3469
23	3493	3516	3540	3564	3588	3612	3636	3660	3685	3709
24	3733	3758	3782	3807	3832	3856	3881	3906	3931	3956
25	3981	4007	4032	4057	4083	4108	4134	4160	4185	4211
26	4237	4263	4289	4315	4341	4368	4394	4420	4447	4473
27	4500	4527	4553	4580	4607	4634	4661	4688	4716	4743
28	4770	4798	4825	4853	4881	4908	4936	4964	4992	5020
29	5048	5076	5105	5133	5161	5190	5218	5247	5276	5304

Sta.	Grade per Station.								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
3	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7
4	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6
5	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
6	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
7	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
8	0.8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2
9	0.9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1
10	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
11	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
12	1.2	2.4	3.6	4.8	6.0	7.2	8.4	9.6	10.8
13	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
14	1.4	2.8	4.2	5.6	7.0	8.4	9.8	11.2	12.6
15	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0	13.5
16	1.6	3.2	4.8	6.4	8.0	9.6	11.2	12.8	14.4
17	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
18	1.8	3.6	5.4	7.2	9.0	10.8	12.6	14.4	16.2
19	1.9	3.8	5.7	7.6	9.5	11.4	13.3	15.2	17.1
20	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0
21	2.1	4.2	6.3	8.4	10.5	12.6	14.7	16.8	18.9
22	2.2	4.4	6.6	8.8	11.0	13.2	15.4	17.6	19.8
23	2.3	4.6	6.9	9.2	11.5	13.8	16.1	18.4	20.7
24	2.4	4.8	7.2	9.6	12.0	14.4	16.8	19.2	21.6
25	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0	22.5
26	2.6	5.2	7.8	10.4	13.0	15.6	18.2	20.8	23.4
27	2.7	5.4	8.1	10.8	13.5	16.2	18.9	21.6	24.3
28	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
29	2.9	5.8	8.7	11.6	14.5	17.4	20.3	23.2	26.1
30	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0
31	3.1	6.2	9.3	12.4	15.5	18.6	21.7	24.8	27.9
32	3.2	6.4	9.6	12.8	16.0	19.2	22.4	25.6	28.8
33	3.3	6.6	9.9	13.2	16.5	19.8	23.1	26.4	29.7
34	3.4	6.8	10.2	13.6	17.0	20.4	23.8	27.2	30.6
35	3.5	7.0	10.5	14.0	17.5	21.0	24.5	28.0	31.5
36	3.6	7.2	10.8	14.4	18.0	21.6	25.2	28.8	32.4
37	3.7	7.4	11.1	14.8	18.5	22.2	25.9	29.6	33.3
38	3.8	7.6	11.4	15.2	19.0	22.8	26.6	30.4	34.2
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40.5
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0

TABLE XI.—CORRECTION FOR THE EARTH'S CURVATURE AND FOR REFRACTION.

Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*	Dist. *D.*	Cor. *E.*
300	.002	1800	.066	3300	.223	4800	.472
400	.003	1900	.074	3400	.237	4900	.492
500	.005	2000	.082	3500	.251	5000	.512
600	.007	2100	.090	3600	.266	5100	.533
700	.010	2200	.099	3700	.281	5200	.554
800	.013	2300	.108	3800	.296	1 mile	.571
900	.017	2400	.118	3900	.312	2 miles	2.285
1000	.020	2500	.128	4000	.328	3 "	5.142
1100	.025	2600	.139	4100	.345	4 "	9.142
1200	.030	2700	.149	4200	.362	5 "	14.284
1300	.035	2800	.161	4300	.379	6 "	20.568
1400	.040	2900	.172	4400	.397	7 "	27.996
1500	.046	3000	.184	4500	.415	8 "	36.566
1600	.052	3100	.197	4600	.434	9 "	46.279
1700	.059	3200	.210	4700	.453	10 "	57.135

TABLE XII.—ELEVATION OF OUTER RAIL ON CURVES.

Degree *D.*	Velocity in Miles per Hour.									
	10	15	20	25	30	35	40	45	50	60
1	.006	.013	.023	.036	.051	.070	.091	.115	.143	.205
2	.011	.026	.046	.071	.103	.140	.182	.231	.285	.410
3	.017	.038	.068	.107	.154	.209	.274	.346	.428	.616
4	.023	.051	.091	.142	.205	.279	.365	.462	.570	.821
5	.029	.064	.114	.178	.257	.349	.456	.577	.712	
6	.034	.077	.137	.214	.308	.419	.547	.693	.855	
7	.040	.090	.160	.249	.359	.489	.638	.808	.997	
8	.046	.103	.182	.285	.410	.559	.730	.923		
9	.051	.115	.205	.321	.462	.628	.821			
10	.057	.128	.228	.356	.513	.698	.912			
12	.068	.153	.274	.428	.616	.838				
14	.080	.180	.319	.499	.718	.978				
16	.091	.205	.365	.570	.821					
18	.103	.231	.410	.641	.923					
20	.114	.257	.456	.712						
25	.143	.321	.570	.891						
30	.171	.385	.684							
35	.200	.449	.798							
40	.228	.513	.912							
50	.285	.641								

′	0°		1°		2°		3°		4°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.00000	One.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	One.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	One.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	One.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	One.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	One.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	One.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	One.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	One.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	One.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	One.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	89°		88°		87°		86°		85°		

′	5°		6°		7°		8°		9°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	84°		83°		82°		81°		80°		

′	10°		11°		12°		13°		14°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97371	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.96837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97998	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.23345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	
′	79°		78°		77°		76°		75°		′

′	15°		16°		17°		18°		19°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.96347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.95231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.93580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	74°		73°		72°		71°		70°		

TABLE XX.—SINES AND COSINES.

′	20°		21°		22°		23°		24°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92444	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.92410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.38510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	69°		68°		67°		66°		65°		

′	25°		26°		27°		28°		29°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.89739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.49141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	
′	64°		63°		62°		61°		60°		′

′	30°		31°		32°		33°		34°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	59°		58°		57°		56°		55°		

′	35°		36°		37°		38°		39°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.63158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	40
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	
	54°		53°		52°		51°		50°		

′	40°		41°		42°		43°		44°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76236	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66038	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	35
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74352	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	49°		48°		47°		46°		45°		

′	0°		1°		2°		3°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.87	.01804	55.4115	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		

′	0°		1°		2°		3°		4°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.00000	One.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	One.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	One.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	One.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	One.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	One.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	One.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	One.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	One.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	One.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	One.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	89°		88°		87°		86°		85°		

′	8°		9°		10°		11°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	6.91174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18384	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91236	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5 86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	81°		80°		79°		78°		

′	12°		13°		14°		15°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3 60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27921	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03076	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	77°		76°		75°		74°		

′	16°		17°		18°		19°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2 87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	73°		72°		71°		70°		

′	20°		21°		22°		23°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2 67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2 48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2 24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	69°		68°		67°		66°		

′	24°		25°		26°		27°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45538	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97681	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52985	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	65°		64°		63°		62°		

′	28°		29°		30°		31°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56501	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58905	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76629	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1 61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	61°		60°		59°		58°		

′	32°		33°		34°		35°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65024	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62953	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72167	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	57°		56°		55°		54°		

′	40°		41°		42°		43°		44°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.68772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73688	.68878	.72497	70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	35
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	73570	.69004	.72377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69067	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74352	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	49°		48°		47°		46°		45°		

′	0°		1°		2°		3°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7036	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02473	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00815	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04833	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6833	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		

′	4°		5°		6°		7°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33155	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
28	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	9.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11 5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	85°		84°		83°		82°		

′	8°		9°		10°		11°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	6.91174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18384	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91236	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	81°		80°		79°		78°		

′	12°		13°		14°		15°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3 60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27921	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03076	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	77°		76°		75°		74°		

′	16°		17°		18°		19°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2 87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35020	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35118	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	.33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94591	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	.36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	73°		72°		71°		70°		

′	20°		21°		22°		23°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	.36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37123	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2 67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38863	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	2.62103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2 48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2 24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	
′	69°		68°		67°		66°		′

′	24°		25°		26°		27°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45538	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97681	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52985	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	65°		64°		63°		62°		

′	28°		29°		30°		31°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56501	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58905	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76629	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1 61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	
′	61°		60°		59°		58°		′

′	32°		33°		34°		35°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65024	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62953	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50797	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45139	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72167	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	57°		56°		55°		54°		

′	36°		37°		38°		39°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.73457	1.36134	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	53°		52°		51°		50°		

′	40°		41°		42°		43°		′
	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1 11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		

′	44°		′	′	44°		′	′	44°		′
	Tang	Cotang			Tang	Cotang			Tang	Cotang	
0	.96569	1.03553	60	20	.97700	1.02355	40	40	.98843	1.01170	20
1	.96625	1.03493	59	21	.97756	1.02295	39	41	.98901	1.01112	19
2	.96681	1.03433	58	22	.97813	1.02236	38	42	.98958	1.01053	18
3	.96738	1.03372	57	23	.97870	1.02176	37	43	.99016	1.00994	17
4	.96794	1.03312	56	24	.97927	1.02117	36	44	.99073	1.00935	16
5	.96850	1.03252	55	25	.97984	1.02057	35	45	.99131	1.00876	15
6	.96907	1.03192	54	26	.98041	1.01998	34	46	.99189	1.00818	14
7	.96963	1.03132	53	27	.98098	1.01939	33	47	.99247	1.00759	13
8	.97020	1.03072	52	28	.98155	1.01879	32	48	.99304	1.00701	12
9	.97076	1.03012	51	29	.98213	1.01820	31	49	.99362	1.00642	11
10	.97133	1.02952	50	30	.98270	1.01761	30	50	.99420	1.00583	10
11	.97189	1.02892	49	31	.98327	1.01702	29	51	.99478	1.00525	9
12	.97246	1.02832	48	32	.98384	1.01642	28	52	.99536	1.00467	8
13	.97302	1.02772	47	33	.98441	1.01583	27	53	.99594	1.00408	7
14	.97359	1.02713	46	34	.98499	1.01524	26	54	.99652	1.00350	6
15	.97416	1.02653	45	35	.98556	1.01465	25	55	.99710	1.00291	5
16	.97472	1.02593	44	36	.98613	1.01406	24	56	.99768	1.00233	4
17	.97529	1.02533	43	37	.98671	1.01347	23	57	.99826	1.00175	3
18	.97586	1.02474	42	38	.98728	1.01288	22	58	.99884	1.00116	2
19	.97643	1.02414	41	39	.98786	1.01229	21	59	.99942	1.00058	1
20	.97700	1.02355	40	40	.98843	1.01170	20	60	1.00000	1.00000	0
′	Cotang	Tang	′	′	Cotang	Tang	′	′	Cotang	Tang	′
	45°				45°				45°		

′	0°		1°		2°		3°		′
	Vers.	**Exsec.**	**Vers.**	**Exsec.**	**Vers.**	**Exsec.**	**Vers.**	**Exsec.**	
0	.00000	.00000	.00015	.00015	.00061	.00061	.00137	.00137	**0**
1	.00000	.00000	.00016	.00016	.00062	.00062	.00139	00139	**1**
2	.00000	.00000	.00016	.00016	.00063	.00063	.00140	.00140	**2**
3	.00000	.00000	.00017	.00017	.00064	.00064	.00142	.00142	**3**
4	.00000	.00000	.00017	.00017	.00065	.00065	.00143	.00143	**4**
5	.00000	.00000	.00018	.00018	.00066	.00066	.00145	.00145	**5**
6	.00000	.00000	.00018	.00018	.00067	.00067	.00146	.00147	**6**
7	.00000	.00000	.00019	.00019	.00068	.00068	.00148	.00148	**7**
8	.00000	.00000	.00020	.00020	.00069	.00069	.00150	.00150	**8**
9	.00000	.00000	.00020	.00020	.00070	.00070	.00151	.00151	**9**
10	.00000	.00000	.00021	.00021	.00071	.00072	.00153	.00153	**10**
11	.00001	.00001	.00021	.00021	.00073	.00073	.00154	.00155	**11**
12	.00001	.00001	.00022	.00022	.00074	.00074	.00156	.00156	**12**
13	.00001	.00001	.00023	.00023	.00075	.00075	.00158	.00158	**13**
14	.00001	.00001	.00023	.00023	.00076	.00076	.00159	.00159	**14**
15	.00001	.00001	.00024	.00024	.00077	.00077	.00161	.00161	**15**
16	.00001	.00001	.00024	.00024	.00078	.00078	.00162	.00163	**16**
17	.00001	.00001	.00025	.00025	.00079	.00079	.00164	.00164	**17**
18	.00001	.00001	.00026	.00026	.00081	.00081	.00166	.00166	**18**
19	.00002	.00002	.00026	.00026	.00082	.00082	.00168	.00168	**19**
20	.00002	.00002	.00027	.00027	.00083	.00083	.00169	.00169	**20**
21	.00002	.00002	.00028	.00028	.00084	.00084	.00171	.00171	**21**
22	.00002	.00002	.00028	.00028	.00085	.00085	.00173	.00173	**22**
23	.00002	.00002	.00029	.00029	.00087	.00087	.00174	.00175	**23**
24	.00002	.00002	.00030	.00030	.00088	.00088	.00176	.00176	**24**
25	.00003	.00003	.00031	.00031	.00089	.00089	.00178	.00178	**25**
26	.00003	.00003	.00031	.00031	.00090	.00090	.00179	.00180	**26**
27	.00003	.00003	.00032	.00032	.00091	.00091	.00181	.00182	**27**
28	.00003	.00003	.00033	.00033	.00093	.00093	.00183	.00183	**28**
29	.00004	.00004	.00034	.00034	.00094	.00094	.00185	.00185	**29**
30	.00004	.00004	.00034	.00034	.00095	.00095	.00187	.00187	**30**
31	.00004	.00004	.00035	.00035	.00096	.00097	.00188	.00189	**31**
32	.00004	.00004	.00036	.00036	.00098	.00098	.00190	.00190	**32**
33	.00005	.00005	.00037	.00037	.00099	.00099	.00192	.00192	**33**
34	.00005	.00005	.00037	.00037	.00100	.00100	.00194	.00194	**34**
35	.00005	.00005	.00038	.00038	.00102	.00102	.00196	.00196	**35**
36	.00005	.00005	.00039	.00039	.00103	.00103	.00197	.00198	**36**
37	.00006	.00006	.00040	.00040	.00104	.00104	.00199	.00200	**37**
38	.00006	.00006	.00041	.00041	.00106	.00106	.00201	.00201	**38**
39	.00006	.00006	.00041	.00041	.00107	.00107	.00203	.00203	**39**
40	.00007	.00007	.00042	.00042	.00108	.00108	.00205	.00205	**40**
41	.00007	.00007	.00043	.00043	.00110	.00110	.00207	.00207	**41**
42	.00007	.00007	.00014	.00044	.00111	.00111	.00208	.00209	**42**
43	.00008	.00008	.00045	.00045	.00112	.00113	.00210	.00211	**43**
44	.00008	.00008	.00046	.00046	.00114	.00114	.00212	.00213	**44**
45	.00009	.00009	.00047	.00047	.00115	.00115	.00214	.00215	**45**
46	.00009	.00009	.00048	.00048	.00117	.00117	.00216	.00216	**46**
47	.00009	.00009	.00048	.00048	.00118	.00118	.00218	.00218	**47**
48	.00010	.00010	.00049	.00049	.00119	.00120	.00220	.00220	**48**
49	.00010	.00010	.00050	.00050	.00121	.00121	.00222	.00222	**49**
50	.00011	.00011	.00051	.00051	.00122	.00122	.00224	.00224	**50**
51	.00011	.00011	.00052	.00052	.00124	.00124	.00226	.00226	**51**
52	.00011	.00011	.00053	.00053	.00125	.00125	.00228	.00228	**52**
53	.00012	.00012	.00054	.00054	.00127	.00127	.00230	.00230	**53**
54	.00012	.00012	.00055	.00055	.00128	.00128	.00232	.00232	**54**
55	.00013	.00013	.00056	.00056	.00130	.00130	.00234	.00234	**55**
56	.00013	.00013	.00057	.00057	.00131	.00131	.00236	.00236	**56**
57	.00014	.00014	.00058	.00058	.00133	.00133	.00238	.00238	**57**
58	.00014	.00014	.00059	.00059	.00134	.00134	.00240	.00240	**58**
59	.00015	.00015	.00060	.00060	.00136	.00136	.00242	.00242	**59**
60	.00015	.00015	.00061	.00061	.00137	.00137	.00244	.00244	

′	4°		5°		6°		7°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.00244	.00244	.00381	.00382	.00548	.00551	.00745	.00751	0
1	.00246	.00246	.00383	.00385	.00551	.00554	.00749	.00755	1
2	.00248	.00248	.00386	.00387	.00554	.00557	.00752	.00758	2
3	.00250	.00250	.00388	.00390	.00557	.00560	.00756	.00762	3
4	.00252	.00252	.00391	.00392	.00560	.00563	.00760	.00765	4
5	.00254	.00254	.00393	.00395	.00563	.00566	.00763	.00769	5
6	.00256	.00257	.00396	.00397	.00566	.00569	.00767	.00773	6
7	.00258	.00259	.00398	.00400	.00569	.00573	.00770	.00776	7
8	.00260	.00261	.00401	.00403	.00572	.00576	.00774	.00780	8
9	.00262	.00263	.00404	.00405	.00576	.00579	.00778	.00784	9
10	.00264	.00265	.00406	.00408	.00579	.00582	.00781	.00787	10
11	.00266	.00267	.00409	.00411	00582	.00585	.00785	.00791	11
12	.00269	.00269	.00412	.00413	.00585	.00588	.00789	.00795	12
13	.00271	.00271	.00414	.00416	.00588	.00592	.00792	.00799	13
14	.00273	.00274	.00417	.00419	.00591	.00595	.00796	.00802	14
15	.00275	.00276	.00420	.00421	00594	.00598	.00800	.00806	15
16	.00277	.00278	.00422	.00424	.00598	.00601	.00803	.00810	16
17	.00279	.00280	.00425	.00427	.00601	.00604	.00807	.00813	17
18	.00281	.00282	.00428	.00429	.00604	.00608	.00811	.00817	18
19	.00284	.00284	.00430	.00432	.00607	.00611	.00814	.00821	19
20	.00286	.00287	.00433	.00435	.00610	.00614	.00818	.00825	20
21	.00288	.00289	.00436	.00438	.00614	.00617	.00822	.00828	21
22	.00290	.00291	.00438	.00440	.00617	.00621	.00825	.00832	22
23	.00293	.00293	.00441	.00443	.00620	.00624	.00829	.00836	23
24	.00295	.00296	.00444	.00446	.00623	.00627	.00833	.00840	24
25	.00297	.00298	.00447	.00449	.00626	.00630	.00837	.00844	25
26	.00299	.00300	.00449	.00451	.00630	.00634	.00840	.00848	26
27	.00301	.00302	.00452	.00454	.00633	.00637	.00844	.00851	27
28	.00304	.00305	.00455	.00457	.00636	.00640	.00848	.00855	28
29	.00306	.00307	.00458	.00460	.00640	.00644	.00852	.00859	29
30	.00308	.00309	.00460	.00463	.00643	.00647	.00856	.00863	30
31	.00311	.00312	.00463	.00465	.00646	.00650	.00859	.00867	31
32	.00313	.00314	.00466	.00468	.00649	.00654	.00863	.00871	32
33	.00315	.00316	.00469	.00471	.00653	.00657	.00867	.00875	33
34	.00317	.00318	.00472	.00474	.00656	.00660	.00871	.00878	34
35	.00320	.00321	.00474	.00477	.00659	.00664	.00875	.00882	35
36	.00322	.00323	.00477	.00480	.00663	.00667	.00878	.00886	36
37	.00324	.00326	.00480	.00482	.00666	.00671	.00882	.00890	37
38	.00327	.00328	.00483	.00485	.00669	.00674	.00886	.00894	38
39	.00329	.00330	.00486	.00488	.00673	.00677	.00890	.00898	39
40	.00332	.00333	.00489	.00491	.00676	.00681	.00894	.00902	40
41	.00334	.00335	.00492	.00494	.00680	.00684	.00898	.00906	41
42	.00336	.00337	.00494	.00497	.00683	.00688	.00902	.00910	42
43	.00339	.00340	.00497	.00500	.00686	.00691	.00906	.00914	43
44	.00341	.00342	.00500	.00503	.00690	.00695	.00909	.00918	44
45	.00343	.00345	.00503	.00506	.00693	.00698	00913	.00922	45
46	.00346	.00347	.00506	.00509	.00697	.00701	00917	.00926	46
47	.00348	.00350	.00509	.00512	.00700	.00705	.00921	.00930	47
48	.00351	.00352	.00512	.00515	.00703	.00708	.00925	.00934	48
49	.00353	.00354	.00515	.00518	.00707	.00712	00929	.00938	49
50	.00356	.00357	.00518	.00521	.00710	.00715	.00933	.00942	50
51	.00358	.00359	.00521	.00524	.00714	.00719	.00937	.00946	51
52	.00361	.00362	.00524	.00527	.00717	.00722	.00941	.00950	52
53	.00363	.00364	.00527	.00530	.00721	.00726	.00945	.00954	53
54	.00365	.00367	.00530	.00533	.00724	.00730	.00949	.00958	54
55	.00368	.00369	.00533	.00536	.00728	.00733	.00953	.00962	55
56	.00370	.00372	.00536	.00539	.00731	.00737	.00957	.00966	56
57	.00373	.00374	.00539	.00542	.00735	.00740	.00961	.00970	57
58	.00375	.00377	.00542	.00545	.00738	.00744	.00965	.00975	58
59	.00378	.00379	.00545	.00548	.00742	.00747	.00969	.00979	59
60	.00381	.00382	.00548	.00551	.00745	.00751	.00973	.00983	60

′	8°		9°		10°		11°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.00973	.00983	.01231	.01247	.01519	.01543	.01837	.01872	0
1	.00977	.00987	.01236	.01251	.01524	.01548	.01843	.01877	1
2	.00981	.00991	.01240	.01256	.01529	.01553	.01848	.01883	2
3	.00985	.00995	.01245	.01261	.01534	.01558	.01854	.01889	3
4	.00989	.00999	.01249	.01265	.01540	.01564	.01860	.01895	4
5	.00994	.01004	.01254	.01270	.01545	.01569	.01865	.01901	5
6	.00998	.01008	.01259	.01275	.01550	.01574	.01871	.01906	6
7	.01002	.01012	.01263	.01279	.01555	.01579	.01876	.01912	7
8	.01006	.01016	.01268	.01284	.01560	.01585	.01882	.01918	8
9	.01010	.01020	.01272	.01289	.01565	.01590	.01888	.01924	9
10	.01014	.01024	.01277	.01294	.01570	.01595	.01893	.01930	10
11	.01018	.01029	.01282	.01298	.01575	.01601	.01899	.01936	11
12	.01022	.01033	.01286	.01303	.01580	.01606	.01904	.01941	12
13	.01027	.01037	.01291	.01308	.01586	.01611	.01910	.01947	13
14	.01031	.01041	.01296	.01313	.01591	.01616	.01916	.01953	14
15	.01035	.01046	.01300	.01318	.01596	.01622	.01921	.01959	15
16	.01039	.01050	.01305	.01322	.01601	.01627	.01927	.01965	16
17	.01043	.01054	.01310	.01327	.01606	.01633	.01933	.01971	17
18	.01047	.01059	.01314	.01332	.01612	.01638	.01939	.01977	18
19	.01052	.01063	.01319	.01337	.01617	.01643	.01944	.01983	19
20	.01056	.01067	.01324	.01342	.01622	.01649	.01950	.01989	20
21	.01060	.01071	.01329	.01346	.01627	.01654	.01956	.01995	21
22	.01064	.01076	.01333	.01351	.01632	.01659	.01961	.02001	22
23	.01069	.01080	.01338	.01356	.01638	.01665	.01967	.02007	23
24	.01073	.01084	.01343	.01361	.01643	.01670	.01973	.02013	24
25	.01077	.01089	.01348	.01366	.01648	.01676	.01979	.02019	25
26	.01081	.01093	.01352	.01371	.01653	.01681	.01984	.02025	26
27	.01086	.01097	.01357	.01376	.01659	.01687	.01990	.02031	27
28	.01090	.01102	.01362	.01381	.01664	.01692	.01996	.02037	28
29	.01094	.01106	.01367	.01386	.01669	.01698	.02002	.02043	29
30	.01098	.01111	.01371	.01391	.01675	.01703	.02008	.02049	30
31	.01103	.01115	.01376	.01395	.01680	.01709	.02013	.02055	31
32	.01107	.01119	.01381	.01400	.01685	.01714	.02019	.02061	32
33	.01111	.01124	.01386	.01405	.01690	.01720	.02025	.02067	33
34	.01116	.01128	.01391	.01410	.01696	.01725	.02031	.02073	34
35	.01120	.01133	.01396	.01415	.01701	.01731	.02037	.02079	35
36	.01124	.01137	.01400	.01420	.01706	.01736	.02042	.02085	36
37	.01129	.01142	.01405	.01425	.01712	.01742	.02048	.02091	37
38	.01133	.01146	.01410	.01430	.01717	.01747	.02054	.02097	38
39	.01137	.01151	.01415	.01435	.01723	.01753	.02060	.02103	39
40	.01142	.01155	.01420	.01440	.01728	.01758	.02066	.02110	40
41	.01146	.01160	.01425	.01445	.01733	.01764	.02072	.02116	41
42	.01151	.01164	.01430	.01450	.01739	.01769	.02078	.02122	42
43	.01155	.01169	.01435	.01455	.01744	.01775	.02084	.02128	43
44	.01159	.01173	.01439	.01461	.01750	.01781	.02090	.02134	44
45	.01164	.01178	.01444	.01466	.01755	.01786	.02095	.02140	45
46	.01168	.01182	.01449	.01471	.01760	.01792	.02101	.02146	46
47	.01173	.01187	.01454	.01476	.01766	.01798	.02107	.02153	47
48	.01177	.01191	.01459	.01481	.01771	.01803	.02113	.02159	48
49	.01182	.01196	.01464	.01486	.01777	.01809	.02119	.02165	49
50	.01186	.01200	.01469	.01491	.01782	.01815	.02125	.02171	50
51	.01191	.01205	.01474	.01496	.01788	.01820	.02131	.02178	51
52	.01195	.01209	.01479	.01501	.01793	.01826	.02137	.02184	52
53	.01200	.01214	.01484	.01506	.01799	.01832	.02143	.02190	53
54	.01204	.01219	.01489	.01512	.01804	.01837	.02149	.02196	54
55	.01209	.01223	.01494	.01517	.01810	.01843	.02155	.02203	55
56	.01213	.01228	.01499	.01522	.01815	.01849	.02161	.02209	56
57	.01218	.01233	.01504	.01527	.01821	.01854	.02167	.02215	57
58	.01222	.01237	.01509	.01532	.01826	.01860	.02173	.02221	58
59	.01227	.01242	.01514	.01537	.01832	.01866	.02179	.02228	59
60	.01231	.01247	.01519	.01543	.01837	.01872	.02185	.02234	60

′	12°		13°		14°		15°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.02185	.02234	.02563	.02630	.02970	.03061	.03407	.03528	0
1	.02191	.02240	.02570	.02637	.02977	.03069	.03415	.03536	1
2	.02197	.02247	.02576	.02644	.02985	.03076	.03422	.03544	2
3	.02203	.02253	.02583	.02651	.02992	.03084	.03430	.03552	3
4	.02210	.02259	.02589	.02658	.02999	.03091	.03438	.03560	4
5	.02216	.02266	.02596	.02665	.03006	.03099	.03445	.03568	5
6	.02222	.02272	.02602	.02672	.03013	.03106	.03453	.03576	6
7	.02228	.02279	.02609	.02679	.03020	.03114	.03460	.03584	7
8	.02234	.02285	.02616	.02686	.03027	.03121	.03468	.03592	8
9	.02240	.02291	.02622	.02693	.03034	.03129	.03476	.03601	9
10	.02246	.02298	.02629	.02700	.03041	.03137	.03483	.03609	10
11	.02252	.02304	.02635	.02707	.03048	.03144	.03491	.03617	11
12	.02258	.02311	.02642	.02714	.03055	.03152	.03498	.03625	12
13	.02265	.02317	.02649	.02721	.03063	.03159	.03506	.03633	13
14	.02271	.02323	.02655	.02728	.03070	.03167	.03514	.03642	14
15	.02277	.02330	.02662	.02735	.03077	.03175	.03521	.03650	15
16	.02283	.02336	.02669	.02742	.03084	.03182	.03529	.03658	16
17	.02289	.02343	.02675	.02749	.03091	.03190	.03537	.03666	17
18	.02295	.02349	.02682	.02756	.03098	.03198	.03544	.03674	18
19	.02302	.02356	.02689	.02763	.03106	.03205	.03552	.03683	19
20	.02308	.02362	.02696	.02770	.03113	.03213	.03560	.03691	20
21	.02314	.02369	.02702	.02777	.03120	.03221	.03567	.03699	21
22	.02320	.02375	.02709	.02784	.03127	.03228	.03575	.03708	22
23	.02327	.02382	.02716	.02791	.03134	.03236	.03583	.03716	23
24	.02333	.02388	.02722	.02799	.03142	.03244	.03590	.03724	24
25	.02339	.02395	.02729	.02806	.03149	.03251	.03598	.03732	25
26	.02345	.02402	.02736	.02813	.03156	.03259	.03606	.03741	26
27	.02352	.02408	.02743	.02820	.03163	.03267	.03614	.03749	27
28	.02358	.02415	.02749	.02827	.03171	.03275	.03621	.03758	28
29	.02364	.02421	.02756	.02834	.03178	.03282	.03629	.03766	29
30	.02370	.02428	.02763	.02842	.03185	.03290	.03637	.03774	30
31	.02377	.02435	.02770	.02849	.03193	.03298	.03645	.03783	31
32	.02383	.02441	.02777	.02856	.03200	.03306	.03653	.03791	32
33	.02389	.02448	.02783	.02863	.03207	.03313	.03660	.03799	33
34	.02396	.02454	.02790	.02870	.03214	.03321	.03668	.03808	34
35	.02402	.02461	.02797	.02878	.03222	.03329	.03676	.03816	35
36	.02408	.02468	.02804	.02885	.03229	.03337	.03684	.03825	36
37	.02415	.02474	.02811	.02892	.03236	.03345	.03692	.03833	37
38	.02421	.02481	.02818	.02899	.03244	.03353	.03699	.03842	38
39	.02427	.02488	.02824	.02907	.03251	.03360	.03707	.03850	39
40	.02434	.02494	.02831	.02914	.03258	.03368	.03715	.03858	40
41	.02440	.02501	.02838	.02921	.03266	.03376	.03723	.03867	41
42	.02447	.02508	.02845	.02928	.03273	.03384	.03731	.03875	42
43	.02453	.02515	.02852	.02936	.03281	.03392	.03739	.03884	43
44	.02459	.02521	.02859	.02943	.03288	.03400	.03747	.03892	44
45	.02466	.02528	.02866	.02950	.03295	.03408	.03754	.03901	45
46	.02472	.02535	.02873	.02958	.03303	.03416	.03762	.03909	46
47	.02479	.02542	.02880	.02965	.03310	.03424	.03770	.03918	47
48	.02485	.02548	.02887	.02972	.03318	.03432	.03778	.03927	48
49	.02492	.02555	.02894	.02980	.03325	.03439	.03786	.03935	49
50	.02498	.02562	.02900	.02987	.03333	.03447	.03794	.03944	50
51	.02504	.02569	.02907	.02994	.03340	.03455	.03802	.03952	51
52	.02511	.02576	.02914	.03002	.03347	.03463	.03810	.03961	52
53	.02517	.02582	.02921	.03009	.03355	.03471	.03818	.03969	53
54	.02524	.02589	.02928	.03017	.03362	.03479	.03826	.03978	54
55	.02530	.02596	.02935	.03024	.03370	.03487	.03834	.03987	55
56	.02537	.02603	.02942	.03032	.03377	.03495	.03842	.03995	56
57	.02543	.02610	.02949	.03039	.03385	.03503	.03850	.04004	57
58	.02550	.02617	.02956	.03046	.03392	.03512	.03858	.04013	58
59	.02556	.02624	.02963	.03054	.03400	.03520	.03866	.04021	59
60	.02563	.02630	.02970	.03061	.03407	.03528	.03874	.04030	60

′	16°		17°		18°		19°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.03874	.04030	.04370	.04569	.04894	.05146	.05448	.05762	0
1	.03882	.04039	.04378	.04578	.04903	.05156	.05458	.05773	1
2	.03890	.04047	.04387	.04588	.04912	.05166	.05467	.05783	2
3	.03898	.04056	.04395	.04597	.04921	.05176	.05477	.05794	3
4	.03906	.04065	.04404	.04606	.04930	.05186	.05486	.05805	4
5	.03914	.04073	.04412	.04616	.04939	.05196	.05496	.05815	5
6	.03922	.04082	.04421	.04625	.04948	.05206	.05505	.05826	6
7	.03930	.04091	.04429	.04635	.04957	.05216	.05515	.05836	7
8	.03938	.04100	.04438	.04644	.04967	.05226	.05524	.05847	8
9	.03946	.04108	.04446	.04653	.04976	.05236	.05534	.05858	9
10	.03954	.04117	.04455	.04663	.04985	.05246	.05543	.05869	10
11	.03963	.04126	.04464	.04672	.04994	.05256	.05553	.05879	11
12	.03971	.04135	.04472	.04682	.05003	.05266	.05562	.05890	12
13	.03979	.04144	.04481	.04691	.05012	.05276	.05572	.05901	13
14	.03987	.04152	.04489	.04700	.05021	.05286	.05582	.05911	14
15	.03995	.04161	.04498	.04710	.05030	.05297	.05591	.05922	15
16	.04003	.04170	.04507	.04719	.05039	.05307	.05601	.05933	16
17	.04011	.04179	.04515	.04729	.05048	.05317	.05610	.05944	17
18	.04019	.04188	.04524	.04738	.05057	.05327	.05620	.05955	18
19	.04028	.04197	.04533	.04748	.05067	.05337	.05630	.05965	19
20	.04036	.04206	.04541	.04757	.05076	.05347	.05639	.05976	20
21	.04044	.04214	.04550	.04767	.05085	.05357	.05649	.05987	21
22	.04052	.04223	.04559	.04776	.05094	.05367	.05658	.05998	22
23	.04060	.04232	.04567	.04786	.05103	.05378	.05668	.06009	23
24	.04069	.04241	.04576	.04795	.05112	.05388	.05678	.06020	24
25	.04077	.04250	.04585	.04805	.05122	.05398	.05687	.06030	25
26	.04085	.04259	.04593	.04815	.05131	.05408	.05697	.06041	26
27	.04093	.04268	.04602	.04824	.05140	.05418	.05707	.06052	27
28	.04102	.04277	.04611	.04834	.05149	.05429	.05716	.06063	28
29	.04110	.04286	.04620	.04843	.05158	.05439	.05726	.06074	29
30	.04118	.04295	.04628	.04853	.05168	.05449	.05736	.06085	30
31	.04126	.04304	.04637	.04863	.05177	.05460	.05746	.06096	31
32	.04135	.04313	.04646	.04872	.05186	.05470	.05755	.06107	32
33	.04143	.04322	.04655	.04882	.05195	.05480	.05765	.06118	33
34	.04151	.04331	.04663	.04891	.05205	.05490	.05775	.06129	34
35	.04159	.04340	.04672	.04901	.05214	.05501	.05785	.06140	35
36	.04168	.04349	.04681	.04911	.05223	.05511	.05794	.06151	36
37	.04176	.04358	.04690	.04920	.05232	.05521	.05804	.06162	37
38	.04184	.04367	.04699	.04930	.05242	.05532	.05814	.06173	38
39	.04193	.04376	.04707	.04940	.05251	.05542	.05824	.06184	39
40	.04201	.04385	.04716	.04950	.05260	.05552	.05833	.06195	40
41	.04209	.04394	.04725	.04959	.05270	.05563	.05843	.06206	41
42	.04218	.04403	.04734	.04969	.05279	.05573	.05853	.06217	42
43	.04226	.04413	.04743	.04979	.05288	.05584	.05863	.06228	43
44	.04234	.04422	.04752	.04989	.05298	.05594	.05873	.06239	44
45	.04243	.04431	.04760	.04998	.05307	.05604	.05882	.06250	45
46	04251	.04440	.04769	.05008	.05316	.05615	.05892	.06261	46
47	.04260	.04449	.04778	.05018	.05326	.05625	.05902	.06272	47
48	.04268	.04458	.04787	.05028	.05335	.05636	.05912	.06283	48
49	.04276	.04468	.04796	.05038	.05344	.05646	.05922	.06295	49
50	.04285	.04477	.04805	.05047	.05354	.05657	.05932	.06306	50
51	.04293	.04486	.04814	.05057	.05363	.05667	.05942	.06317	51
52	.04302	.04495	.04823	.05067	.05373	.05678	.05951	.06328	52
53	.04310	.04504	.04832	.05077	.05382	.05688	.05961	.06339	53
54	.04319	.04514	.04841	.05087	.05391	.05699	.05971	.06350	54
55	.04327	.04523	.04850	.05097	.05401	.05709	.05981	.06362	55
56	.04336	.04532	.04858	.05107	.05410	.05720	.05991	.06373	56
57	.04344	.04541	.04867	.05116	.05420	.05730	.06001	.06384	57
58	.04353	.04551	.04876	.05126	.05429	.05741	.06011	.06395	58
59	.04361	.04560	.04885	.05136	.05439	.05751	.06021	.06407	59
60	.04370	.04569	.04894	.05146	.05448	.05762	.06031	.06418	60

′	20°		21°		22°		23°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.06031	.06418	.06642	.07115	.07282	.07853	.07950	.08636	0
1	.06041	.06429	.06652	.07126	.07293	.07866	.07961	.08649	1
2	.06051	.06440	.06663	.07138	.07303	.07879	.07972	.08663	2
3	.06061	.06452	.06673	.07150	.07314	.07892	.07984	.08676	3
4	.06071	.06463	.06684	.07162	.07325	.07904	.07995	.08690	4
5	.06081	.06474	.06694	.07174	.07336	.07917	.08006	.08703	5
6	.06091	.06486	.06705	.07186	.07347	.07930	.08018	.08717	6
7	.06101	.06497	.06715	.07199	.07358	07943	.08029	.08730	7
8	.06111	.06508	.06726	.07211	.07369	.07955	.08041	.08744	8
9	.06121	.06520	.06736	.07223	.07380	.07968	.08052	.08757	9
10	.06131	.06531	.06747	.07235	.07391	.07981	.08064	.08771	10
11	.06141	.06542	.06757	.07247	.07402	.07994	.08075	.08784	11
12	.06151	.06554	.06768	.07259	.07413	.08006	.08086	.08798	12
13	.06161	.06565	.06778	.07271	.07424	.08019	.08098	.08811	13
14	.06171	.06577	.06789	.07283	.07435	.08032	.08109	.08825	14
15	.06181	.06588	.06799	.07295	.07446	.08045	.08121	.08839	15
16	.06191	.06600	.06810	.07307	.07457	.08058	.08132	.08852	16
17	.06201	.06611	.06820	.07320	.07468	.08071	.08144	.08866	17
18	.06211	.06622	.06831	.07332	.07479	.08084	.08155	.08880	18
19	.06221	.06634	.06841	.07344	.07490	.08097	.08167	.08893	19
20	.06231	.06645	.06852	.07356	.07501	.08109	.08178	.08907	20
21	.06241	.06657	.06863	.07368	.07512	.08122	.08190	.08921	21
22	.06252	.06668	.06873	.07380	.07523	.08135	.08201	.08934	22
23	.06262	.06680	.06884	.07393	.07534	.08148	.08213	.08948	23
24	.06272	.06691	.06894	.07405	.07545	.08161	.08225	.08962	24
25	.06282	.06703	.06905	.07417	.07556	.08174	.08236	.08975	25
26	.06292	.06715	.06916	.07429	.07568	.08187	.08248	.08989	26
27	.06302	.06726	.06926	.07442	.07579	.08200	.08259	.09003	27
28	.06312	.06738	.06937	.07454	.07590	.08213	.08271	.09017	28
29	.06323	.06749	.06948	.07466	.07601	.08226	.08282	.09030	29
30	.06333	.06761	.06958	.07479	.07612	.08239	.08294	.09044	30
31	.06343	.06773	.06969	.07491	.07623	.08252	.08306	.09058	31
32	.06353	.06784	.06980	.07503	.07634	.08265	.08317	.09072	32
33	.06363	.06796	.06990	.07516	.07645	.08278	.08329	.09086	33
34	.06374	.06807	.07001	.07528	.07657	.08291	.08340	.09099	34
35	.06384	.06819	.07012	.07540	.07668	.08305	.08352	.09113	35
36	.06394	.06831	.07022	.07553	.07679	.08318	.08364	.09127	36
37	.06404	.06843	.07033	.07565	.07690	.08331	.08375	.09141	37
38	.06415	.06854	.07044	.07578	.07701	.08344	.08387	.09155	38
39	.06425	.06866	.07055	.07590	.07713	.08357	.08399	.09169	39
40	.06435	.06878	.07065	.07602	.07724	.08370	.08410	.09183	40
41	.06445	.06889	.07076	.07615	.07735	.08383	.08422	.09197	41
42	.06456	.06901	.07087	.07627	.07746	.08397	.08434	.09211	42
43	.06466	.06913	.07098	.07640	.07757	.08410	.08445	.09224	43
44	.06476	.06925	.07108	.07652	.07769	.08423	.08457	.09238	44
45	.06486	.06936	.07119	.07665	.07780	.08436	.08469	.09252	45
46	.06497	.06948	.07130	.07677	.07791	.08449	.08481	.09266	46
47	.06507	.06960	.07141	.07690	.07802	.08463	.08492	.09280	47
48	.06517	.06972	.07151	.07702	.07814	.08476	.08504	.09294	48
49	.06528	.06984	.07162	.07715	.07825	.08489	.08516	.09308	49
50	.06538	.06995	.07173	.07727	.07836	.08503	.08528	.09323	50
51	.06548	.07007	.07184	.07740	.07848	.08516	.08539	.09337	51
52	.06559	.07019	.07195	.07752	.07859	.08529	.08551	.09351	52
53	.06569	.07031	.07206	.07765	.07870	.08542	.08563	.09365	53
54	.06580	.07043	.07216	.07778	.07881	.08556	.08575	.09379	54
55	.06590	.07055	.07227	.07790	.07893	.08569	.08586	.09393	55
56	.06600	.07067	.07238	.07803	.07904	.08582	.08598	.09407	56
57	.06611	.07079	.07249	.07816	.07915	.08596	.08610	.09421	57
58	.06621	.07091	.07260	.07828	.07927	.08609	.08622	.09435	58
59	.06632	.07103	.07271	.07841	.07938	.08623	.08634	.09449	59
60	.06642	.07115	.07282	.07853	.07950	.08636	.08645	.09464	60

′	24°		25°		26°		27°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.08645	.09464	.09369	.10338	.10121	.11260	.10899	.12233	0
1	.08657	.09478	.09382	.10353	.10133	.11276	.10913	.12249	1
2	.08669	.09492	.09394	.10368	.10146	.11292	.10926	.12266	2
3	.08681	.09506	.09406	.10383	.10159	.11308	.10939	.12283	3
4	.08693	.09520	.09418	.10398	.10172	.11323	.10952	.12299	4
5	.08705	.09535	.09431	.10413	.10184	.11339	.10965	.12316	5
6	.08717	.09549	.09443	.10428	.10197	.11355	.10979	.12333	6
7	.08728	.09563	.09455	.10443	.10210	.11371	.10992	.12349	7
8	.08740	.09577	.09468	.10458	.10223	.11387	.11005	.12366	8
9	.08752	.09592	.09480	.10473	.10236	.11403	.11019	.12383	9
10	.08764	.09606	.09493	.10488	.10248	.11419	.11032	.12400	10
11	.08776	.09620	.09505	.10503	.10261	.11435	.11045	.12416	11
12	.08788	.09635	.09517	.10518	.10274	.11451	.11058	.12433	12
13	.08800	.09649	.09530	.10533	.10287	.11467	.11072	.12450	13
14	.08812	.09663	.09542	.10549	.10300	.11483	.11085	.12467	14
15	.08824	.09678	.09554	.10564	.10313	.11499	.11098	.12484	15
16	.08836	.09692	.09567	.10579	.10326	.11515	.11112	.12501	16
17	.08848	.09707	.09579	.10594	.10338	.11531	.11125	.12518	17
18	.08860	.09721	.09592	.10609	.10351	.11547	.11138	.12534	18
19	.08872	.09735	.09604	.10625	.10364	.11563	.11152	.12551	19
20	.08884	.09750	.09617	.10640	.10377	.11579	.11165	.12568	20
21	.08896	.09764	.09629	.10655	.10390	.11595	.11178	.12585	21
22	.08908	.09779	.09642	.10670	.10403	.11611	.11192	.12602	22
23	.08920	.09793	.09654	.10686	.10416	.11627	.11205	.12619	23
24	.08932	.09808	.09666	.10701	.10429	.11643	.11218	.12636	24
25	.08944	.09822	.09679	.10716	.10442	.11659	.11232	.12653	25
26	.08956	.09837	.09691	.10731	.10455	.11675	.11245	.12670	26
27	.08968	.09851	.09704	.10747	.10468	.11691	.11259	.12687	27
28	.08980	.09866	.09716	.10762	.10481	.11708	.11272	.12704	28
29	.08992	.09880	.09729	.10777	.10494	.11724	.11285	.12721	29
30	.09004	.09895	.09741	.10793	.10507	.11740	.11299	.12738	30
31	.09016	.09909	.09754	.10808	.10520	.11756	.11312	.12755	31
32	.09028	.09924	.09767	.10824	.10533	.11772	.11326	.12772	32
33	.09040	.09939	.09779	.10839	.10546	.11789	.11339	.12789	33
34	.09052	.09953	.09792	.10854	.10559	.11805	.11353	.12807	34
35	.09064	.09968	.09804	.10870	.10572	.11821	.11366	.12824	35
36	.09076	.09982	.09817	.10885	.10585	.11838	.11380	.12841	36
37	.09089	.09997	.09829	.10901	.10598	.11854	.11393	.12858	37
38	.09101	.10012	.09842	.10916	.10611	.11870	.11407	.12875	38
39	.09113	.10026	.09854	.10932	.10624	.11886	.11420	.12892	39
40	.09125	.10041	.09867	.10947	.10637	.11903	.11434	.12910	40
41	.09137	.10055	.09880	.10963	.10650	.11919	.11447	.12927	41
42	.09149	.10071	.09892	.10978	.10663	.11936	.11461	.12944	42
43	.09161	.10085	.09005	.10994	.10676	.11952	.11474	.12961	43
44	.09174	.10100	.09918	.11009	.10689	.11968	.11488	.12979	44
45	.09186	.10115	.09930	.11025	.10702	.11985	.11501	.12996	45
46	.09198	.10130	.09943	.11041	.10715	.12001	.11515	.13013	46
47	.09210	.10144	.09955	.11056	.10728	.12018	.11528	.13031	47
48	.09222	.10159	.09968	.11072	.10741	.12034	.11542	.13048	48
49	.09234	.10174	.09981	.11087	.10755	.12051	.11555	.13065	49
50	.09247	.10189	.09993	.11103	.10768	.12067	.11569	.13083	50
51	.09259	.10204	.10006	.11119	.10781	.12084	.11583	.13100	51
52	.09271	.10218	.10019	.11134	.10794	.12100	.11596	.13117	52
53	.09283	.10233	.10032	.11150	.10807	.12117	.11610	.13135	53
54	.09296	.10248	.10044	.11166	.10820	.12133	.11623	.13152	54
55	.09308	.10263	.10057	.11181	.10833	.12150	.11637	.13170	55
56	.09320	.10278	.10070	.11197	.10847	.12166	.11651	.13187	56
57	.09332	.10293	.10082	.11213	.10860	.12183	.11664	.13205	57
58	.09345	.10308	.10095	.11229	.10873	.12199	.11678	.13222	58
59	.09357	.10323	.10108	.11244	.10886	.12216	.11692	.13240	
60	.09369	.10338	.10121	.11260	.10899	.12233	.11705	.1325	

′	28°		29°		30°		31°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.11705	.13257	.12538	.14335	.13397	.15470	.14283	.16663	0
1	.11719	.13275	.12552	.14354	.13412	.15489	.14298	.16684	1
2	.11733	.13292	.12566	.14372	.13427	.15509	.14313	.16704	2
3	.11746	.13310	.12580	.14391	.13441	.15528	.14328	.16725	3
4	.11760	.13327	.12595	.14409	.13456	.15548	.14343	.16745	4
5	.11774	.13345	.12609	.14428	.13470	.15567	.14358	.16766	5
6	.11787	.13362	.12623	.14446	.13485	.15587	.14373	.16786	6
7	.11801	.13380	.12637	.14465	.13499	.15606	.14388	.16806	7
8	.11815	.13398	.12651	.14483	.13514	.15626	.14403	.16827	8
9	.11828	.13415	.12665	.14502	.13529	.15645	.14418	.16848	9
10	.11842	.13433	.12679	.14521	.13543	.15665	.14433	.16868	10
11	.11856	.13451	.12694	.14539	.13558	.15684	.14449	.16889	11
12	.11870	.13468	.12708	.14558	.13573	.15704	.14464	.16909	12
13	.11883	.13486	.12722	.14576	.13587	.15724	.14479	.16930	13
14	.11897	.13504	.12736	.14595	.13602	.15743	.14494	.16950	14
15	.11911	.13521	.12750	.14614	.13616	.15763	.14509	.16971	15
16	.11925	.13539	.12765	.14632	.13631	.15782	.14524	.16992	16
17	.11938	.13557	.12779	.14651	.13646	.15802	.14539	.17012	17
18	.11952	.13575	.12793	.14670	.13660	.15822	.14554	.17033	18
19	.11966	.13593	.12807	.14689	.13675	.15841	.14569	.17054	19
20	.11980	.13610	.12822	.14707	.13690	.15861	.14584	.17075	20
21	.11994	.13628	.12836	.14726	.13705	.15881	.14599	.17095	21
22	.12007	.13646	.12850	.14745	.13719	.15901	.14615	.17116	22
23	.12021	.13664	.12864	.14764	.13734	.15920	.14630	.17137	23
24	.12035	.13682	.12879	.14782	.13749	.15940	.14645	.17158	24
25	.12049	.13700	.12893	.14801	.13763	.15960	.14660	.17178	25
26	.12063	.13718	.12907	.14820	.13778	.15980	.14675	.17199	26
27	.12077	.13735	.12921	.14839	.13793	.16000	.14690	.17220	27
28	.12091	.13753	.12936	.14858	.13808	.16019	.14706	.17241	28
29	.12104	.13771	.12950	.14877	.13822	.16039	.14721	.17262	29
30	.12118	.13789	.12964	.14896	.13837	.16059	.14736	.17283	30
31	.12132	.13807	.12979	.14914	.13852	.16079	.14751	.17304	31
32	.12146	.13825	.12993	.14933	.13867	.16099	.14766	.17325	32
33	.12160	.13843	.13007	.14952	.13881	.16119	.14782	.17346	33
34	.12174	.13861	.13022	.14971	.13896	.16139	.14797	.17367	34
35	.12188	.13879	.13036	.14990	.13911	.16159	.14812	.17388	35
36	.12202	.13897	.13051	.15009	.13926	.16179	.14827	.17409	36
37	.12216	.13916	.13065	.15028	.13941	.16199	.14843	.17430	37
38	.12230	.13934	.13079	.15047	.13955	.16219	.14858	.17451	38
39	.12244	.13952	.13094	.15066	.13970	.16239	.14873	.17472	39
40	.12257	.13970	.13108	.15085	.13985	.16259	.14888	.17493	40
41	.12271	.13988	.13122	.15105	.14000	.16279	.14904	.17514	41
42	.12285	.14006	.13137	.15124	.14015	.16299	.14919	.17535	42
43	.12299	.14024	.13151	.15143	.14030	.16319	.14934	.17556	43
44	.12313	.14042	.13166	.15162	.14044	.16339	.14949	.17577	44
45	.12327	.14061	.13180	.15181	.14059	.16359	.14965	.17598	45
46	.12341	.14079	.13195	.15200	.14074	.16380	.14980	.17620	46
47	.12355	.14097	.13209	.15219	.14089	.16400	.14995	.17641	47
48	.12369	.14115	.13223	.15239	.14104	.16420	.15011	.17662	48
49	.12383	.14134	.13238	.15258	.14119	.16440	.15026	.17683	49
50	.12397	.14152	.13252	.15277	.14134	.16460	.15041	.17704	50
51	.12411	.14170	.13267	.15296	.14149	.16481	.15057	.17726	51
52	.12425	.14188	.13281	.15315	.14164	.16501	.15072	.17747	52
53	.12439	.14207	.13296	.15335	.14179	.16521	.15087	.17768	53
54	.12454	.14225	.13310	.15354	.14194	.16541	.15103	.17790	54
55	.12468	.14243	.13325	.15373	.14208	.16562	.15118	.17811	55
56	.12482	.14262	.13339	.15393	.14223	.16582	.15134	.17832	56
57	.12496	.14280	.13354	.15412	.14238	.16602	.15149	.17854	57
58	.12510	.14299	.13368	.15431	.14253	.16623	.15164	.17875	58
59	.12524	.14317	.13383	.15451	.14268	.16643	.15180	.17896	59
	.12538	.14335	.13397	.15470	.14283	.16663	.15195	.17918	60

′	32°		33°		34°		35°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.15195	.17918	.16133	.19236	.17096	.20622	.18085	.22077	0
1	.15211	.17939	.16149	.19259	.17113	.20645	.18101	.22102	1
2	.15226	.17961	.16165	.19281	.17129	.20669	.18118	.22127	2
3	.15241	.17982	.16181	.19304	.17145	.20693	.18135	.22152	3
4	.15257	.18004	.16196	.19327	.17161	.20717	.18152	.22177	4
5	.15272	.18025	.16212	.19349	.17178	.20740	.18168	.22202	5
6	.15288	.18047	.16228	.19372	.17194	.20764	.18185	.22227	6
7	.15303	.18068	.16244	.19394	.17210	.20788	.18202	.22252	7
8	.15319	.18090	.16260	.19417	.17227	.20812	.18218	.22277	8
9	.15334	.18111	.16276	.19440	.17243	.20836	.18235	.22302	9
10	.15350	.18133	.16292	.19463	.17259	.20859	.18252	.22327	10
11	.15365	.18155	.16308	.19485	.17276	.20883	.18269	.22352	11
12	.15381	.18176	.16324	.19508	.17292	.20907	.18286	.22377	12
13	.15396	.18198	.16340	.19531	.17308	.20931	.18302	.22402	13
14	.15412	.18220	.16355	.19554	.17325	.20955	.18319	.22428	14
15	.15427	.18241	.16371	.19576	.17341	.20979	.18336	.22453	15
16	.15443	.18263	.16387	.19599	.17357	.21003	.18353	.22478	16
17	.15458	.18285	.16403	.19622	.17374	.21027	.18369	.22503	17
18	.15474	.18307	.16419	.19645	.17390	.21051	.18386	.22528	18
19	.15489	.18328	.16435	.19668	.17407	.21075	.18403	.22554	19
20	.15505	.18350	.16451	.19691	.17423	.21099	.18420	.22579	20
21	.15520	.18372	.16467	.19713	.17439	.21123	.18437	.22604	21
22	.15536	.18394	.16483	.19736	.17456	.21147	.18454	.22629	22
23	.15552	.18416	.16499	.19759	.17472	.21171	.18470	.22655	23
24	.15567	.18437	.16515	.19782	.17489	.21195	.18487	.22680	24
25	.15583	.18459	.16531	.19805	.17505	.21220	.18504	.22706	25
26	.15598	.18481	.16547	.19828	.17522	.21244	.18521	.22731	26
27	.15614	.18503	.16563	.19851	.17538	.21268	.18538	.22756	27
28	.15630	.18525	.16579	.19874	.17554	.21292	.18555	.22782	28
29	.15645	.18547	.16595	.19897	.17571	.21316	.18572	.22807	29
30	.15661	.18569	.16611	.19920	.17587	.21341	.18588	.22833	30
31	.15676	.18591	.16627	.19944	.17604	.21365	.18605	.22858	31
32	.15692	.18613	.16644	.19967	.17620	.21389	.18622	.22884	32
33	.15708	.18635	.16660	.19990	.17637	.21414	.18639	.22909	33
34	.15723	.18657	.16676	.20013	.17653	.21438	.18656	.22935	34
35	.15739	.18679	.16692	.20036	.17670	.21462	.18673	.22960	35
36	.15755	.18701	.16708	.20059	.17686	.21487	.18690	.22986	36
37	.15770	.18723	.16724	.20083	.17703	.21511	.18707	.23012	37
38	.15786	.18745	.16740	.20106	.17719	.21535	.18724	.23037	38
39	.15802	.18767	.16756	.20129	.17736	.21560	.18741	.23063	39
40	.15818	.18790	.16772	.20152	.17752	.21584	.18758	.23089	40
41	.15833	.18812	.16788	.20176	.17769	.21609	.18775	.23114	41
42	.15849	.18834	.16805	.20199	.17786	.21633	.18792	.23140	42
43	.15865	.18856	.16821	.20222	.17802	.21658	.18809	.23166	43
44	.15880	.18878	.16837	.20246	.17819	.21682	.18826	.23192	44
45	.15896	.18901	.16853	.20269	.17835	.21707	.18843	.23217	45
46	.15912	.18923	.16869	.20292	.17852	.21731	.18860	.23243	46
47	.15928	.18945	.16885	.20316	.17868	.21756	.18877	.23269	47
48	.15943	.18967	.16902	.20339	.17885	.21781	.18894	.23295	48
49	.15959	.18990	.16918	.20363	.17902	.21805	.18911	.23321	49
50	.15975	.19012	.16934	.20386	.17918	.21830	.18928	.23347	50
51	.15991	.19034	.16950	.20410	.17935	.21855	.18945	.23373	51
52	.16006	.19057	.16966	.20433	.17952	.21879	.18962	.23399	52
53	.16022	.19079	.16983	.20457	.17968	.21904	.18979	.23424	53
54	.16038	.19102	.16999	.20480	.17985	.21929	.18996	.23450	54
55	.16054	.19124	.17015	.20504	.18001	.21953	.19013	.23476	55
56	.16070	.19146	.17031	.20527	.18018	.21978	.19030	.23502	56
57	.16085	.19169	.17047	.20551	.18035	.22003	.19047	.23529	57
58	.16101	.19191	.17064	.20575	.18051	.22028	.19064	.23555	58
59	.16117	.19214	.17080	.20598	.18068	.22053	.19081	.23581	59
60	.16133	.19236	.17096	.20622	.18085	.22077	.19098	.23607	60

TABLE XXII.—VERSINES AND EXSECANTS.

′	36°		37°		38°		39°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.19098	.23607	.20136	.25214	.21199	.26902	.22285	.28676	0
1	.19115	.23633	.20154	.25241	.21217	.26931	.22304	.28706	1
2	.19133	.23659	.20171	.25269	.21235	.26960	.22322	.28737	2
3	.19150	.23685	.20189	.25296	.21253	.26988	.22340	.28767	3
4	.19167	.23711	.20207	.25324	.21271	.27017	.22359	.28797	4
5	.19184	.23738	.20224	.25351	.21289	.27046	.22377	.28828	5
6	.19201	.23764	.20242	.25379	.21307	.27075	.22395	.28858	6
7	.19218	.23790	.20259	.25406	.21324	.27104	.22414	.28889	7
8	.19235	.23816	.20277	.25434	.21342	.27133	.22432	.28919	8
9	.19252	.23843	.20294	.25462	.21360	.27162	.22450	.28950	9
10	.19270	.23869	.20312	.25489	.21378	.27191	.22469	.28980	10
11	.19287	.23895	.20329	.25517	.21396	.27221	.22487	.29011	11
12	.19304	.23922	.20347	.25545	.21414	.27250	.22506	.29042	12
13	.19321	.23948	.20365	.25572	.21432	.27279	.22524	.29072	13
14	.19338	.23975	.20382	.25600	.21450	.27308	.22542	.29103	14
15	.19356	.24001	.20400	.25628	.21468	.27337	.22561	.29133	15
16	.19373	.24028	.20417	.25656	.21486	.27366	.22579	.29164	16
17	.19390	.24054	.20435	.25683	.21504	.27396	.22598	.29195	17
18	.19407	.24081	.20453	.25711	.21522	.27425	.22616	.29226	18
19	.19424	.24107	.20470	.25739	.21540	.27454	.22634	.29256	19
20	.19442	.24134	.20488	.25767	.21558	.27483	.22653	.29287	20
21	.19459	.24160	.20506	.25795	.21576	.27513	.22671	.29318	21
22	.19476	.24187	.20523	.25823	.21595	.27542	.22690	.29349	22
23	.19493	.24213	.20541	.25851	.21613	.27572	.22708	.29380	23
24	.19511	.24240	.20559	.25879	.21631	.27601	.22727	.29411	24
25	.19528	.24267	.20576	.25907	.21649	.27630	.22745	.29442	25
26	.19545	.24293	.20594	.25935	.21667	.27660	.22764	.29473	26
27	.19562	.24320	.20612	.25963	.21685	.27689	.22782	.29504	27
28	.19580	.24347	.20629	.25991	.21703	.27719	.22801	.29535	28
29	.19597	.24373	.20647	.26019	.21721	.27748	.22819	.29566	29
30	.19614	.24400	.20665	.26047	.21739	.27778	.22838	.29597	30
31	.19632	.24427	.20682	.26075	.21757	.27807	.22856	.29628	31
32	.19649	.24454	.20700	.26104	.21775	.27837	.22875	.29659	32
33	.19666	.24481	.20718	.26132	.21794	.27867	.22893	.29690	33
34	.19684	.24508	.20736	.26160	.21812	.27896	.22912	.29721	34
35	.19701	.24534	.20753	.26188	.21830	.27926	.22930	.29752	35
36	.19718	.24561	.20771	.26216	.21848	.27956	.22949	.29784	36
37	.19736	.24588	.20789	.26245	.21866	.27985	.22967	.29815	37
38	.19753	.24615	.20807	.26273	.21884	.28015	.22986	.29846	38
39	.19770	.24642	.20824	.26301	.21902	.28045	.23004	.29877	39
40	.19788	.24669	.20842	.26330	.21921	.28075	.23023	.29909	40
41	.19805	.24696	.20860	.26358	.21939	.28105	.23041	.29940	41
42	.19822	.24723	.20878	.26387	.21957	.28134	.23060	.29971	42
43	.19840	.24750	.20895	.26415	.21975	.28164	.23079	.30003	43
44	.19857	.24777	.20913	.26443	.21993	.28194	.23097	.30034	44
45	.19875	.24804	.20931	.26472	.22012	.28224	.23116	.30066	45
46	.19892	.24832	.20949	.26500	.22030	.28254	.23134	.30097	46
47	.19909	.24859	.20967	.26529	.22048	.28284	.23153	.30129	47
48	.19927	.24886	.20985	.26557	.22066	.28314	.23172	.30160	48
49	.19944	.24913	.21002	.26586	.22084	.28344	.23190	.30192	49
50	.19962	.24940	.21020	.26615	.22103	.28374	.23209	.30223	50
51	.19979	.24967	.21038	.26643	.22121	.28404	.23228	.30255	51
52	.19997	.24995	.21056	.26672	.22139	.28434	.23246	.30287	52
53	.20014	.25022	.21074	.26701	.22157	.28464	.23265	.30318	53
54	.20032	.25049	.21092	.26729	.22176	.28495	.23283	.30350	54
55	.20049	.25077	.21109	.26758	.22194	.28525	.23302	.30382	55
56	.20066	.25104	.21127	.26787	.22212	.28555	.23321	.30413	56
57	.20084	.25131	.21145	.26815	.22231	.28585	.23339	.30445	57
58	.20101	.25159	.21163	.26844	.22249	.28615	.23358	.30477	58
59	.20119	.25186	.21181	.26873	.22267	.28646	.23377	.30509	59
60	.20136	.25214	.21199	.26902	.22285	.28676	.23396	.30541	60

′	40°		41°		42°		43°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.23396	.30541	.24529	.32501	.25686	.34563	.26865	.36733	0
1	.23414	.30573	.24548	.32535	.25705	.34599	.26884	.36770	1
2	.23433	.30605	.24567	.32568	.25724	.34634	.26904	.36807	2
3	.23452	.30636	.24586	.32602	.25744	.34669	.26924	.36844	3
4	.23470	.30668	.24605	.32636	.25763	.34704	.26944	.36881	4
5	.23489	.30700	.24625	.32669	.25783	.34740	.26964	.36919	5
6	.23508	.30732	.24644	.32703	.25802	.34775	.26984	.36956	6
7	.23527	.30764	.24663	.32737	.25822	.34811	.27004	.36993	7
8	.23545	.30796	.24682	.32770	.25841	.34846	.27024	.37030	8
9	.23564	.30829	.24701	.32804	.25861	.34882	.27043	.37068	9
10	.23583	.30861	.24720	.32838	.25880	.34917	.27063	.37105	10
11	.23602	.30893	.24739	.32872	.25900	.34953	.27083	.37143	11
12	.23620	.30925	.24759	.32905	.25920	.34988	.27103	.37180	12
13	.23639	.30957	.24778	.32939	.25939	.35024	.27123	.37218	13
14	.23658	.30989	.24797	.32973	.25959	.35060	.27143	.37255	14
15	.23677	.31022	.24816	.33007	.25978	.35095	.27163	.37293	15
16	.23696	.31054	.24835	.33041	.25998	.35131	.27183	.37330	16
17	.23714	.31086	.24854	.33075	.26017	.35167	.27203	.37368	17
18	.23733	.31119	.24874	.33109	.26037	.35203	.27223	.37406	18
19	.23752	.31151	.24893	.33143	.26056	.35238	.27243	.37443	19
20	.23771	.31183	.24912	.33177	.26076	.35274	.27263	.37481	20
21	.23790	.31216	.24931	.33211	.26096	.35310	.27283	.37519	21
22	.23808	.31248	.24950	.33245	.26115	.35346	.27303	.37556	22
23	.23827	.31281	.24970	.33279	.26135	.35382	.27323	.37594	23
24	.23846	.31313	.24989	.33314	.26154	.35418	.27343	.37632	24
25	.23865	.31346	.25008	.33348	.26174	.35454	.27363	.37670	25
26	.23884	.31378	.25027	.33382	.26194	.35490	.27383	.37708	26
27	.23903	.31411	.25047	.33416	.26213	.35526	.27403	.37746	27
28	.23922	.31443	.25066	.33451	.26233	.35562	.27423	.37784	28
29	.23941	.31476	.25085	.33485	.26253	.35598	.27443	.37822	29
30	.23959	.31509	.25104	.33519	.26272	.35634	.27463	.37860	30
31	.23978	.31541	.25124	.33554	.26292	.35670	.27483	.37898	31
32	.23997	.31574	.25143	.33588	.26312	.35707	.27503	.37936	32
33	.24016	.31607	.25162	.33622	.26331	.35743	.27523	.37974	33
34	.24035	.31640	.25182	.33657	.26351	.35779	.27543	.38012	34
35	.24054	.31672	.25201	.33691	.26371	.35815	.27563	.38051	35
36	.24073	.31705	.25220	.33726	.26390	.35852	.27583	.38089	36
37	.24092	.31738	.25240	.33760	.26410	.35888	.27603	.38127	37
38	.24111	.31771	.25259	.33795	.26430	.35924	.27623	.38165	38
39	.24130	.31804	.25278	.33830	.26449	.35961	.27643	.38204	39
40	.24149	.31837	.25297	.33864	.26469	.35997	.27663	.38242	40
41	.24168	.31870	.25317	.33899	.26489	.36034	.27683	.38280	41
42	.24187	.31903	.25336	.33934	.26509	.36070	.27703	.38319	42
43	.24206	.31936	.25356	.33968	.26528	.36107	.27723	.38357	43
44	.24225	.31969	.25375	.34003	.26548	.36143	.27743	.38396	44
45	.24244	.32002	.25394	.34038	.26568	.36180	.27764	.38434	45
46	.24262	.32035	.25414	.34073	.26588	.36217	.27784	.38473	46
47	.24281	.32068	.25433	.34108	.26607	.36253	.27804	.38512	47
48	.24300	.32101	.25452	.34142	.26627	.36290	.27824	.38550	48
49	.24320	.32134	.25472	.34177	.26647	.36327	.27844	.38589	49
50	.24339	.32168	.25491	.34212	.26667	.36363	.27864	.38628	50
51	.24358	.32201	.25511	.34247	.26686	.36400	.27884	.38666	51
52	.24377	.32234	.25530	.34282	.26706	.36437	.27905	.38705	52
53	.24396	.32267	.25549	.34317	.26726	.36474	.27925	.38744	53
54	.24415	.32301	.25569	.34352	.26746	.36511	.27945	.38783	54
55	.24434	.32334	.25588	.34387	.26766	.36548	.27965	.38822	55
56	.24453	.32368	.25608	.34423	.26785	.36585	.27985	.38860	56
57	.24472	.32401	.25627	.34458	.26805	.36622	.28005	.38899	57
58	.24491	.32434	.25647	.34493	.26825	.36659	.28026	.38938	58
59	.24510	.32468	.25666	.34528	.26845	.36696	.28046	.38977	
60	.24529	.32501	.25686	.34563	.26865	.36733	.28066	.39016	

′	44°		45°		46°		47°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.28066	.39016	.29289	.41421	.30534	.43956	.31800	.46628	0
1	.28086	.39055	.29310	.41463	.30555	.43999	.31821	.46674	1
2	.28106	.39095	.29330	.41504	.30576	.44042	.31843	.46719	2
3	.28127	.39134	.29351	.41545	.30597	.44086	.31864	.46765	3
4	.28147	.39173	.29372	.41586	.30618	.44129	.31885	.46811	4
5	.28167	.39212	.29392	.41627	.30639	.44173	.31907	.46857	5
6	.28187	.39251	.29413	.41669	.30660	.44217	.31928	.46903	6
7	.28208	.39291	.29433	.41710	.30681	.44260	.31949	.46949	7
8	.28228	.39330	.29454	.41752	.30702	.44304	.31971	.46995	8
9	.28248	.39369	.29475	.41793	.30723	.44347	.31992	.47041	9
10	.28268	.39409	.29495	.41835	.30744	.44391	.32013	.47087	10
11	.28289	.39448	.29516	.41876	.30765	.44435	.32035	.47134	11
12	.28309	.39487	.29537	.41918	.30786	.44479	.32056	.47180	12
13	.28329	.39527	.29557	.41959	.30807	.44523	.32077	.47226	13
14	.28350	.39566	.29578	.42001	.30828	.44567	.32099	.47272	14
15	.28370	.39606	.29599	.42042	.30849	.44610	.32120	.47319	15
16	.28390	.39646	.29619	.42084	.30870	.44654	.32141	.47365	16
17	.28410	.39685	.29640	.42126	.30891	.44698	.32163	.47411	17
18	.28431	.39725	.29661	.42168	.30912	.44742	.32184	.47458	18
19	.28451	.39764	.29681	.42210	.30933	.44787	.32205	.47504	19
20	.28471	.39804	.29702	.42251	.30954	.44831	.32227	.47551	20
21	.28492	.39844	.29723	.42293	.30975	.44875	.32248	.47598	21
22	.28512	.39884	.29743	.42335	.30996	.44919	.32270	.47644	22
23	.28532	.39924	.29764	.42377	.31017	.44963	.32291	.47691	23
24	.28553	.39963	.29785	.42419	.31038	.45007	.32312	.47738	24
25	.28573	.40003	.29805	.42461	.31059	.45052	.32334	.47784	25
26	.28593	.40043	.29826	.42503	.31080	.45096	.32355	.47831	26
27	.28614	.40083	.29847	.42545	.31101	.45141	.32377	.47878	27
28	.28634	.40123	.29868	.42587	.31122	.45185	.32398	.47925	28
29	.28655	.40163	.29888	.42630	.31143	.45229	.32420	.47972	29
30	.28675	.40203	.29909	.42672	.31165	.45274	.32441	.48019	30
31	.28695	.40243	.29930	.42714	.31186	.45319	.32462	.48066	31
32	.28716	.40283	.29951	.42756	.31207	.45363	.32484	.48113	32
33	.28736	.40324	.29971	.42799	.31228	.45408	.32505	.48160	33
34	.28757	.40364	.29992	.42841	.31249	.45452	.32527	.48207	34
35	.28777	.40404	.30013	.42883	.31270	.45497	.32548	.48254	35
36	.28797	.40444	.30034	.42926	.31291	.45542	.32570	.48301	36
37	.28818	.40485	.30054	.42968	.31312	.45587	.32591	.48349	37
38	.28838	.40525	.30075	.43011	.31334	.45631	.32613	.48396	38
39	.28859	.40565	.30096	.43053	.31355	.45676	.32634	.48443	39
40	.28879	.40606	.30117	.43096	.31376	.45721	.32656	.48491	40
41	.28900	.40646	.30138	.43139	.31397	.45766	.32677	.48538	41
42	.28920	.40687	.30158	.43181	.31418	.45811	.32699	.48586	42
43	.28941	.40727	.30179	.43224	.31439	.45856	.32720	.48633	43
44	.28961	.40768	.30200	.43267	.31461	.45901	.32742	.48681	44
45	.28981	.40808	.30221	.43310	.31482	.45946	.32763	.48728	45
46	.29002	.40849	.30242	.43352	.31503	.45992	.32785	.48776	46
47	.29022	.40890	.30263	.43395	.31524	.46037	.32806	.48824	47
48	.29043	.40930	.30283	.43438	.31545	.46082	.32828	.48871	48
49	.29063	.40971	.30304	.43481	.31567	.46127	.32849	.48919	49
50	.29084	.41012	.30325	.43524	.31588	.46173	.32871	.48967	50
51	.29104	.41053	.30346	.43567	.31609	.46218	.32893	.49015	51
52	.29125	.41093	.30367	.43610	.31630	.46263	.32914	.49063	52
53	.29145	.41134	.30388	.43653	.31651	.46309	.32936	.49111	53
54	.29166	.41175	.30409	.43696	.31673	.46354	.32957	.49159	54
55	.29187	.41216	.30430	.43739	.31694	.46400	.32979	.49207	55
56	.29207	.41257	.30451	.43783	.31715	.46445	.33001	.49255	56
57	.29228	.41298	.30471	.43826	.31736	.46491	.33022	.49303	57
58	.29248	.41339	.30492	.43869	.31758	.46537	.33044	.49351	58
59	.29269	.41380	.30513	.43912	.31779	.46582	.33065	.49399	59
60	.29289	.41421	.30534	.43956	.31800	.46628	.33087	.49448	60

′	48°		49°		50°		51°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.33087	.49448	.34394	.52425	.35721	.55572	.37068	.58902	0
1	.33109	.49496	.34416	.52476	.35744	.55626	.37091	.58959	1
2	.33130	.49544	.34438	.52527	.35766	.55680	.37113	.59016	2
3	.33152	.49593	.34460	.52579	.35788	.55734	.37136	.59073	3
4	.33173	.49641	.34482	.52630	.35810	.55789	.37158	.59130	4
5	.33195	.49690	.34504	.52681	.35833	.55843	.37181	.59188	5
6	.33217	.49738	.34526	.52732	.35855	.55897	.37204	.59245	6
7	.33238	.49787	.34548	.52784	.35877	.55951	.37226	.59302	7
8	.33260	.49835	.34570	.52835	.35900	.56005	.37249	.59360	8
9	.33282	.49884	.34592	.52886	.35922	.56060	.37272	.59418	9
10	.33303	.49933	.34614	.52938	.35944	.56114	.37294	.59475	10
11	.33325	.49981	.34636	.52989	.35967	.56169	.37317	.59533	11
12	.33347	.50030	.34658	.53041	.35989	.56223	.37340	.59590	12
13	.33368	.50079	.34680	.53092	.36011	.56278	.37362	.59648	13
14	.33390	.50128	.34702	.53144	.36034	.56332	.37385	.59706	14
15	.33412	.50177	.34724	.53196	.36056	.56387	.37408	.59764	15
16	.33434	.50226	.34746	.53247	.36078	.56442	.37430	.59822	16
17	.33455	.50275	.34768	.53299	.36101	.56497	.37453	.59880	17
18	.33477	.50324	.34790	.53351	.36123	.56551	.37476	.59938	18
19	.33499	.50373	.34812	.53403	.36146	.56606	.37498	.59996	19
20	.33520	.50422	.34834	.53455	.36168	.56661	.37521	.60054	20
21	.33542	.50471	.34856	.53507	.36190	.56716	.37544	.60112	21
22	.33564	.50521	.34878	.53559	.36213	.56771	.37567	.60171	22
23	.33586	.50570	.34900	.53611	.36235	.56826	.37589	.60229	23
24	.33607	.50619	.34923	.53663	.36258	.56881	.37612	.60287	24
25	.33629	.50669	.34945	.53715	.36280	.56937	.37635	.60346	25
26	.33651	.50718	.34967	.53768	.36302	.56992	.37658	.60404	26
27	.33673	.50767	.34989	.53820	.36325	.57047	.37680	.60463	27
28	.33694	.50817	.35011	.53872	.36347	.57103	.37703	.60521	28
29	.33716	.50866	.35033	.53924	.36370	.57158	.37726	.60580	29
30	.33738	.50916	.35055	.53977	.36392	.57213	.37749	.60639	30
31	.33760	.50966	.35077	.54029	.36415	.57269	.37771	.60698	31
32	.33782	.51015	.35099	.54082	.36437	.57324	.37794	.60756	32
33	.33803	.51065	.35122	.54134	.36460	.57380	.37817	.60815	33
34	.33825	.51115	.35144	.54187	.36482	.57436	.37840	.60874	34
35	.33847	.51165	.35166	.54240	.36504	.57491	.37862	.60933	35
36	.33869	.51215	.35188	.54292	.36527	.57547	.37885	.60992	36
37	.33891	.51265	.35210	.54345	.36549	.57603	.37908	.61051	37
38	.33912	.51314	.35232	.54398	.36572	.57659	.37931	.61111	38
39	.33934	.51364	.35254	.54451	.36594	.57715	.37954	.61170	39
40	.33956	.51415	.35277	.54504	.36617	.57771	.37976	.61229	40
41	.33978	.51465	.35299	.54557	.36639	.57827	.37999	.61288	41
42	.34000	.51515	.35321	.54610	.36662	.57883	.38022	.61348	42
43	.34022	.51565	.35343	.54663	.36684	.57939	.38045	.61407	43
44	.34044	.51615	.35365	.54716	.36707	.57995	.38068	.61467	44
45	.34065	.51665	.35388	.54769	.36729	.58051	.38091	.61526	45
46	.34087	.51716	.35410	.54822	.36752	.58108	.38113	.61586	46
47	.34109	.51766	.35432	.54876	.36775	.58164	.38136	.61646	47
48	.34131	.51817	.35454	.54929	.36797	.58221	.38159	.61705	48
49	.34153	.51867	.35476	.54982	.36820	.58277	.38182	.61765	49
50	.34175	.51918	.35499	.55036	.36842	.58333	.38205	.61825	50
51	.34197	.51968	.35521	.55089	.36865	.58390	.38228	.61885	51
52	.34219	.52019	.35543	.55143	.36887	.58447	.38251	.61945	52
53	.34241	.52069	.35565	.55196	.36910	.58503	.38274	.62005	53
54	.34262	.52120	.35588	.55250	.36932	.58560	.38296	.62065	54
55	.34284	.52171	.35610	.55303	.36955	.58617	.38319	.62125	55
56	.34306	.52222	.35632	.55357	.36978	.58674	.38342	.62185	56
57	.34328	.52273	.35654	.55411	.37000	.58731	.38365	.62246	57
58	.34350	.52323	.35677	.55465	.37023	.58788	.38388	.62306	58
59	.34372	.52374	.35699	.55518	.37045	.58845	.38411	.62366	
60	.34394	.52425	.35721	.55572	.37068	.58902	.38434	.62427	

′	52°		53°		54°		55°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.38434	.62427	.39819	.66164	.41221	.70130	.42642	.74345	0
1	.38457	.62487	.39842	.66228	.41245	.70198	.42666	.74417	1
2	.38480	.62548	.39865	.66292	.41269	.70267	.42690	.74490	2
3	.38503	.62609	.39888	.66357	.41292	.70335	.42714	.74562	3
4	.38526	.62669	.39911	.66421	.41316	.70403	.42738	.74635	4
5	.38549	.62730	.39935	.66486	.41339	.70472	.42762	.74708	5
6	.38571	.62791	.39958	.66550	.41363	.70540	.42785	.74781	6
7	.38594	.62852	.39981	.66615	.41386	.70609	.42809	.74854	7
8	.38617	.62913	.40005	.66679	.41410	.70677	.42833	.74927	8
9	.38640	.62974	.40028	.66744	.41433	.70746	.42857	.75000	9
10	.38663	.63035	.40051	.66809	.41457	.70815	.42881	.75073	10
11	.38686	.63096	.40074	.66873	.41481	.70884	.42905	.75146	11
12	.38709	.63157	.40098	.66938	.41504	.70953	.42929	.75219	12
13	.38732	.63218	.40121	.67003	.41528	.71022	.42953	.75293	13
14	.38755	.63279	.40144	.67068	.41551	.71091	.42976	.75366	14
15	.38778	.63341	.40168	.67133	.41575	.71160	.43000	.75440	15
16	.38801	.63402	.40191	.67199	.41599	.71229	.43024	.75513	16
17	.38824	.63464	.40214	.67264	.41622	.71298	.43048	.75587	17
18	.38847	.63525	.40237	.67329	.41646	.71368	.43072	.75661	18
19	.38870	.63587	.40261	.67394	.41670	.71437	.43096	.75734	19
20	.38893	.63648	.40284	.67460	.41693	.71506	.43120	.75808	20
21	.38916	.63710	.40307	.67525	.41717	.71576	.43144	.75882	21
22	.38939	.63772	.40331	.67591	.41740	.71646	.43168	.75956	22
23	.38962	.63834	.40354	.67656	.41764	.71715	.43192	.76031	23
24	.38985	.63895	.40378	.67722	.41788	.71785	.43216	.76105	24
25	.39009	.63957	.40401	.67788	.41811	.71855	.43240	.76179	25
26	.39032	.64019	.40424	.67853	.41835	.71925	.43264	.76253	26
27	.39055	.64081	.40448	.67919	.41859	.71995	.43287	.76328	27
28	.39078	.64144	.40471	.67985	.41882	.72065	.43311	.76402	28
29	.39101	.64206	.40494	.68051	.41906	.72135	.43335	.76477	29
30	.39124	.64268	.40518	.68117	.41930	.72205	.43359	.76552	30
31	.39147	.64330	.40541	.68183	.41953	.72275	.43383	.76626	31
32	.39170	.64393	.40565	.68250	.41977	.72346	.43407	.76701	32
33	.39193	.64455	.40588	.68316	.42001	.72416	.43431	.76776	33
34	.39216	.64518	.40611	.68382	.42024	.72487	.43455	.76851	34
35	.39239	.64580	.40635	.68449	.42048	.72557	.43479	.76926	35
36	.39262	.64643	.40658	.68515	.42072	.72628	.43503	.77001	36
37	.39286	.64705	.40682	.68582	.42096	.72698	.43527	.77077	37
38	.39309	.64768	.40705	.68648	.42119	.72769	.43551	.77152	38
39	.39332	.64831	.40728	.68715	.42143	.72840	.43575	.77227	39
40	.39355	.64894	.40752	.68782	.42167	.72911	.43599	.77303	40
41	.39378	.64957	.40775	.68848	.42191	.72982	.43623	.77378	41
42	.39401	.65020	.40799	.68915	.42214	.73053	.43647	.77454	42
43	.39424	.65083	.40822	.68982	.42238	.73124	.43671	.77530	43
44	.39447	.65146	.40846	.69049	.42262	.73195	.43695	.77606	44
45	.39471	.65209	.40869	.69116	.42285	.73267	.43720	.77681	45
46	.39494	.65272	.40893	.69183	.42309	.73338	.43744	.77757	46
47	.39517	.65336	.40916	.69250	.42333	.73409	.43768	.77833	47
48	.39540	.65399	.40939	.69318	.42357	.73481	.43792	.77910	48
49	.39563	.65462	.40963	.69385	.42381	.73552	.43816	.77986	49
50	.39586	.65526	.40986	.69452	.42404	.73624	.43840	.78062	50
51	.39610	.65589	.41010	.69520	.42428	.73696	.43864	.78138	51
52	.39633	.65653	.41033	.69587	.42452	.73768	.43888	.78215	52
53	.39656	.65717	.41057	.69655	.42476	.73840	.43912	.78291	53
54	.39679	.65780	.41080	.69723	.42499	.73911	.43936	.78368	54
55	.39702	.65844	.41104	.69790	.42523	.73983	.43960	.78445	55
56	.39726	.65908	.41127	.69858	.42547	74056	.43984	.78521	56
57	.39749	.65972	.41151	.69926	.42571	.74128	.44008	.78598	57
58	.39772	.66036	.41174	.69994	.42595	.74200	.44032	.78675	58
59	.39795	.66100	.41198	.70062	.42619	.74272	.44057	.78752	59
60	.39819	.66164	.41221	.70130	.42642	.74345	.44081	.78829	60

′	56°		57°		58°		59°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.44081	.78829	.45536	.83608	.47008	.88708	.48496	.94160	0
1	.44105	.78906	.45560	.83690	.47033	.88796	.48521	.94254	1
2	.44129	.78984	.45585	.83773	.47057	.88884	.48546	.94349	2
3	.44153	.79061	.45609	.83855	.47082	.88972	.48571	.94443	3
4	.44177	.79138	.45634	.83938	.47107	.89060	.48596	.94537	4
5	.44201	.79216	.45658	.84020	.47131	.89148	.48621	.94632	5
6	.44225	.79293	.45683	.84103	.47156	.89237	.48646	.94726	6
7	.44250	.79371	.45707	.84186	.47181	.89325	.48671	.94821	7
8	.44274	.79449	.45731	.84269	.47206	.89414	.48696	.94916	8
9	.44298	.79527	.45756	.84352	.47230	.89503	.48721	.95011	9
10	.44322	.79604	.45780	.84435	.47255	.89591	.48746	.95106	10
11	.44346	.79682	.45805	.84518	.47280	.89680	.48771	.95201	11
12	.44370	.79761	.45829	.84601	.47304	.89769	.48796	.95296	12
13	.44395	.79839	.45854	.84685	.47329	.89858	.48821	.95392	13
14	.44419	.79917	.45878	.84768	.47354	.89948	.48846	.95487	14
15	.44443	.79995	.45903	.84852	.47379	.90037	.48871	.95583	15
16	.44467	.80074	.45927	.84935	.47403	.90126	.48896	.95678	16
17	.44491	.80152	.45951	.85019	.47428	.90216	.48921	.95774	17
18	.44516	.80231	.45976	.85103	.47453	.90305	.48946	.95870	18
19	.44540	.80309	.46000	.85187	.47478	.90395	.48971	.95966	19
20	.44564	.80388	.46025	.85271	.47502	.90485	.48996	.96062	20
21	.44588	.80467	.46049	.85355	.47527	.90575	.49021	.96158	21
22	.44612	.80546	.46074	.85439	.47552	.90665	.49046	.96255	22
23	.44637	.80625	.46098	.85523	.47577	.90755	.49071	.96351	23
24	.44661	.80704	.46123	.85608	.47601	.90845	.49096	.96448	24
25	.44685	.80783	.46147	.85692	.47626	.90935	.49121	.96544	25
26	.44709	.80862	.46172	.85777	.47651	.91026	.49146	.96641	26
27	.44734	.80942	.46196	.85861	.47676	.91116	.49171	.96738	27
28	.44758	.81021	.46221	.85946	.47701	.91207	.49196	.96835	28
29	.44782	.81101	.46246	.86031	.47725	.91297	.49221	.96932	29
30	.44806	.81180	.46270	.86116	.47750	.91388	.49246	.97029	30
31	.44831	.81260	.46295	.86201	.47775	.91479	.49271	.97127	31
32	.44855	.81340	.46319	.86286	.47800	.91570	.49296	.97224	32
33	.44879	.81419	.46344	.86371	.47825	.91661	.49321	.97322	33
34	.44903	.81499	.46368	.86457	.47849	.91752	.49346	.97420	34
35	.44928	.81579	.46393	.86542	.47874	.91844	.49372	.97517	35
36	.44952	.81659	.46417	.86627	.47899	.91935	.49397	.97615	36
37	.44976	.81740	.46442	.86713	.47924	.92027	.49422	.97713	37
38	.45001	.81820	.46466	.86799	.47949	.92118	.49447	.97811	38
39	.45025	.81900	.46491	.86885	.47974	.92210	.49472	.97910	39
40	.45049	.81981	.46516	.86990	.47998	.92302	.49497	.98008	40
41	.45073	.82061	.46540	.87056	.48023	.92394	.49522	.98107	41
42	.45098	.82142	.46565	.87142	.48048	.92486	.49547	.98205	42
43	.45122	.82222	.46589	.87229	.48073	.92578	.49572	.98304	43
44	.45146	.82303	.46614	.87315	.48098	.92670	.49597	.98403	44
45	.45171	.82384	.46639	.87401	.48123	.92762	.49623	.98502	45
46	.45195	.82465	.46663	.87488	.48148	.92855	.49648	.98601	46
47	.45219	.82546	.46688	.87574	.48172	.92947	.49673	.98700	47
48	.45244	.82627	.46712	.87661	.48197	.93040	.49698	.98799	48
49	.45268	.82709	.46737	.87748	.48222	.93133	.49723	.98899	49
50	.45292	.82790	.46762	.87834	.48247	.93226	.49748	.98998	50
51	.45317	.82871	.46786	.87921	.48272	.93319	.49773	.99098	51
52	.45341	.82953	.46811	.88008	.48297	.93412	.49799	.99198	52
53	.45365	.83034	.46836	.88095	.48322	.93505	.49824	.99298	53
54	.45390	.83116	.46860	.88183	.48347	.93598	.49849	.99398	54
55	.45414	.83198	.46885	.88270	.48372	.93692	.49874	.99498	55
56	.45439	.83280	.46909	.88357	.48396	.93785	.49899	.99598	56
57	.45463	.83362	.46934	.88445	.48421	.93879	.49924	.99698	57
58	.45487	.83444	.46959	.88532	.48446	.93973	.49950	.99799	58
59	.45512	.83526	.46983	.88620	.48471	.94066	.49975	.99899	59

′	60°		61°		62°		63°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.50000	1.00000	.51519	1.06267	.53053	1.13005	.54601	1.20269	0
1	.50025	1.00101	.51544	1.06375	.53079	1.13122	.54627	1.20395	1
2	.50050	1.00202	.51570	1.06483	.53104	1.13239	.54653	1.20521	2
3	.50076	1.00303	.51595	1.06592	.53130	1.13356	.54679	1.20647	3
4	.50101	1.00404	.51621	1.06701	.53156	1.13473	.54705	1.20773	4
5	.50126	1.00505	.51646	1.06809	.53181	1.13590	.54731	1.20900	5
6	.50151	1.00607	.51672	1.06918	.53207	1.13707	.54757	1.21026	6
7	.50176	1.00708	.51697	1.07027	.53233	1.13825	.54782	1.21153	7
8	.50202	1.00810	.51723	1.07137	.53258	1.13942	.54808	1.21280	8
9	.50227	1.00912	.51748	1.07246	.53284	1.14060	.54834	1.21407	9
10	.50252	1.01014	.51774	1.07356	.53310	1.14178	.54860	1.21535	10
11	.50277	1.01116	.51799	1.07465	.53336	1.14296	.54886	1.21662	11
12	.50303	1.01218	.51825	1.07575	.53361	1.14414	.54912	1.21790	12
13	.50328	1.01320	.51850	1.07685	.53387	1.14533	.54938	1.21918	13
14	.50353	1.01422	.51876	1.07795	.53413	1.14651	.54964	1.22045	14
15	.50378	1.01525	.51901	1.07905	.53439	1.14770	.54990	1.22174	15
16	.50404	1.01628	.51927	1.08015	.53464	1.14889	.55016	1.22302	16
17	.50429	1.01730	.51952	1.08126	.53490	1.15008	.55042	1.22430	17
18	.50454	1.01833	.51978	1.08236	.53516	1.15127	.55068	1.22559	18
19	.50479	1.01936	.52003	1.08347	.53542	1.15246	.55094	1.22688	19
20	.50505	1.02039	.52029	1.08458	.53567	1.15366	.55120	1.22817	20
21	.50530	1.02143	.52054	1.08569	.53593	1.15485	.55146	1.22946	21
22	.50555	1.02246	.52080	1.08680	.53619	1.15605	.55172	1.23075	22
23	.50581	1.02349	.52105	1.08791	.53645	1.15725	.55198	1.23205	23
24	.50606	1.02453	.52131	1.08903	.53670	1.15845	.55224	1.23334	24
25	.50631	1.02557	.52156	1.09014	.53696	1.15965	.55250	1.23464	25
26	.50656	1.02661	.52182	1.09126	.53722	1.16085	.55276	1.23594	26
27	.50682	1.02765	.52207	1.09238	.53748	1.16206	.55302	1.23724	27
28	.50707	1.02869	.52233	1.09350	.53774	1.16326	.55328	1.23855	28
29	.50732	1.02973	.52259	1.09462	.53799	1.16447	.55354	1.23985	29
30	.50758	1.03077	.52284	1.09574	.53825	1.16568	.55380	1.24116	30
31	.50783	1.03182	.52310	1.09686	.53851	1.16689	.55406	1.24247	31
32	.50808	1.03286	.52335	1.09799	.53877	1.16810	.55432	1.24378	32
33	.50834	1.03391	.52361	1.09911	.53903	1.16932	.55458	1.24509	33
34	.50859	1.03496	.52386	1.10024	.53928	1.17053	.55484	1.24640	34
35	.50884	1.03601	.52412	1.10137	.53954	1.17175	.55510	1.24772	35
36	.50910	1.03706	.52438	1.10250	.53980	1.17297	.55536	1.24903	36
37	.50935	1.03811	.52463	1.10363	.54006	1.17419	.55563	1.25035	37
38	.50960	1.03916	.52489	1.10477	.54032	1.17541	.55589	1 25167	38
39	.50986	1.04022	.52514	1.10590	.54058	1.17663	.55615	1.25300	39
40	.51011	1.04128	.52540	1.10704	.54083	1.17786	.55641	1.25432	40
41	.51036	1.04233	.52566	1.10817	.54109	1.17909	.55667	1.25565	41
42	.51062	1.04339	.52591	1.10931	.54135	1.18031	.55693	1.25697	42
43	.51087	1.04445	.52617	1.11045	.54161	1.18154	.55719	1.25830	43
44	.51113	1.04551	.52642	1.11159	.54187	1.18277	.55745	1.25963	44
45	.51138	1.04658	.52668	1.11274	.54213	1.18401	.55771	1.26097	45
46	.51163	1.04764	.52694	1.11388	.54238	1.18524	.55797	1.26230	46
47	.51189	1.04870	.52719	1.11503	.54264	1.18648	.55823	1.26364	47
48	.51214	1.04977	.52745	1.11617	.54290	1.18772	.55849	1.26498	48
49	.51239	1.05084	.52771	1.11732	.54316	1.18895	.55876	1.26632	49
50	.51265	1.05191	.52796	1.11847	.54342	1.19019	.55902	1.26766	50
51	.51290	1.05298	.52822	1.11963	.54368	1.19144	.55928	1.26900	51
52	.51316	1.05405	.52848	1.12078	.54394	1.19268	.55954	1.27035	52
53	.51341	1.05512	.52873	1.12193	.54420	1.19393	.55980	1.27169	53
54	.51366	1.05619	.52899	1.12309	.54446	1.19517	.56006	1.27304	54
55	.51392	1.05727	.52924	1.12425	.54471	1.19642	.56032	1.27439	55
56	.51417	1.05835	.52950	1.12540	.54497	1.19767	.56058	1.27574	56
57	.51443	1.05942	.52976	1.12657	.54523	1.19892	.56084	1.27710	57
58	.51468	1.06050	.53001	1.12773	.54549	1.20018	.56111	1.27845	58
59	.51494	1.06158	.53027	1.12889	.54575	1.20143	.56137	1.27981	59
	.51519	1.06267	.53053	1.13005	.54601	1.20269	.56163	1.28117	60

′	64°		65°		66°		67°		
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.56163	1.28117	.57738	1.36620	.59326	1.45859	.60927	1.55930	0
1	.56189	1.28253	.57765	1.36768	.59353	1.46020	.60954	1.56106	1
2	.56215	1.28390	.57791	1.36916	.59379	1.46181	.60980	1.56282	2
3	.56241	1.28526	.57817	1.37064	.59406	1.46342	.61007	1.56458	3
4	.56267	1.28663	.57844	1.37212	.59433	1.46504	.61034	1.56634	4
5	.56294	1.28800	.57870	1.37361	.59459	1.46665	.61061	1.56811	5
6	.56320	1.28937	.57896	1.37509	.59486	1.46827	.61088	1.56988	6
7	.56346	1.29074	.57923	1.37658	.59512	1.46989	.61114	1.57165	7
8	.56372	1.29211	.57949	1.37808	.59539	1.47152	.61141	1.57342	8
9	.56398	1.29349	.57976	1.37957	.59566	1.47314	.61168	1.57520	9
10	.56425	1.29487	.58002	1.38107	.59592	1.47477	.61195	1.57698	10
11	.56451	1.29625	.58028	1.38256	.59619	1.47640	.61222	1.57876	11
12	.56477	1.29763	.58055	1.38406	.59645	1.47804	.61248	1.58054	12
13	.56503	1.29901	.58081	1.38556	.59672	1.47967	.61275	1.58233	13
14	.56529	1.30040	.58108	1.38707	.59699	1.48131	.61302	1.58412	14
15	.56555	1.30179	.58134	1.38857	.59725	1.48295	.61329	1.58591	15
16	.56582	1.30318	.58160	1.39008	.59752	1.48459	.61356	1.58771	16
17	.56608	1.30457	.58187	1.39159	.59779	1.48624	.61383	1.58950	17
18	.56634	1.30596	.58213	1.39311	.59805	1.48789	.61409	1.59130	18
19	.56660	1.30735	.58240	1.39462	.59832	1.48954	.61436	1.59311	19
20	.56687	1.30875	.58266	1.39614	.59859	1.49119	.61463	1.59491	20
21	.56713	1.31015	.58293	1.39766	.59885	1.49284	.61490	1.59672	21
22	.56739	1.31155	.58319	1.39918	.59912	1.49450	.61517	1.59853	22
23	.56765	1.31295	.58345	1.40070	.59938	1.49616	.61544	1.60035	23
24	.56791	1.31436	.58372	1.40222	.59965	1.49782	.61570	1.60217	24
25	.56818	1.31576	.58398	1.40375	.59992	1.49948	.61597	1.60399	25
26	.56844	1.31717	.58425	1.40528	.60018	1.50115	.61624	1.60581	26
27	.56870	1.31858	.58451	1.40681	.60045	1.50282	.61651	1.60763	27
28	.56896	1.31999	.58478	1.40835	.60072	1.50449	.61678	1.60946	28
29	.56923	1.32140	.58504	1.40988	.60098	1.50617	.61705	1.61129	29
30	.56949	1.32282	.58531	1.41142	.60125	1.50784	.61732	1.61313	30
31	.56975	1.32424	.58557	1.41296	.60152	1.50952	.61759	1.61496	31
32	.57001	1.32566	.58584	1.41450	.60178	1.51120	.61785	1.61680	32
33	.57028	1.32708	.58610	1.41605	.60205	1.51289	.61812	1.61864	33
34	.57054	1.32850	.58637	1.41760	.60232	1.51457	.61839	1.62049	34
35	.57080	1.32993	.58663	1.41914	.60259	1.51626	.61866	1.62234	35
36	.57106	1.33135	.58690	1.42070	.60285	1.51795	.61893	1.62419	36
37	.57133	1.33278	.58716	1.42225	.60312	1.51965	.61920	1.62604	37
38	.57159	1.33422	.58743	1.42380	.60339	1.52134	.61947	1.62790	38
39	.57185	1.33565	.58769	1.42536	.60365	1.52304	.61974	1.62976	39
40	.57212	1.33708	.58796	1.42692	.60392	1.52474	.62001	1.63162	40
41	.57238	1.33852	.58822	1.42848	.60419	1.52645	.62027	1.63348	41
42	.57264	1.33996	.58849	1.43005	.60445	1.52815	.62054	1.63535	42
43	.57291	1.34140	.58875	1.43162	.60472	1.52986	.62081	1.63722	43
44	.57317	1.34284	.58902	1.43318	.60499	1.53157	.62108	1.63909	44
45	.57343	1.34429	.58928	1.43476	.60526	1.53329	.62135	1.64097	45
46	.57369	1.34573	.58955	1.43633	.60552	1.53500	.62162	1.64285	46
47	.57396	1.34718	.58981	1.43790	.60579	1.53672	.62189	1.64473	47
48	.57422	1.34863	.59008	1.43948	.60606	1.53845	.62216	1.64662	48
49	.57448	1.35009	.59034	1.44106	.60633	1.54017	.62243	1.64851	49
50	.57475	1.35154	.59061	1.44264	.60659	1.54190	.62270	1.65040	50
51	.57501	1.35300	.59087	1.44423	.60686	1.54363	.62297	1.65229	51
52	.57527	1.35446	.59114	1.44582	.60713	1.54536	.62324	1.65419	52
53	.57554	1.35592	.59140	1.44741	.60740	1.54709	.62351	1.65609	53
54	.57580	1.35738	.59167	1.44900	.60766	1.54883	.62378	1.65799	54
55	.57606	1.35885	.59194	1.45059	.60793	1.55057	.62405	1.65989	55
56	.57633	1.36031	.59220	1.45219	.60820	1.55231	.62431	1.66180	56
57	.57659	1.36178	.59247	1.45378	.60847	1.55405	.62458	1.66371	57
58	.57685	1.36325	.59273	1.45539	.60873	1.55580	.62485	1.66563	58
59	.57712	1.36473	.59300	1.45699	.60900	1.55755	.62512	1.66755	
60	.57738	1.36620	.59326	1.45859	.60927	1.55930	.62539	1.669	

′	68°		69°		70°		71°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.62539	1.66947	.64163	1.79043	.65798	1.92380	.67443	2.07155	0
1	.62566	1.67139	.64190	1.79254	.65825	1.92614	.67471	2.07415	1
2	.62593	1.67332	.64218	1.79466	.65853	1.92849	.67498	2.07675	2
3	.62620	1.67525	.64245	1.79679	.65880	1.93083	.67526	2.07936	3
4	.62647	1.67718	.64272	1.79891	.65907	1.93318	.67553	2.08197	4
5	.62674	1.67911	.64299	1.80104	.65935	1.93554	.67581	2.08459	5
6	.62701	1.68105	.64326	1.80318	.65962	1.93790	.67608	2.08721	6
7	.62728	1.68299	.64353	1.80531	.65989	1.94026	.67636	2.08983	7
8	.62755	1.68494	.64381	1.80746	.66017	1.94263	.67663	2.09246	8
9	.62782	1.68689	.64408	1.80960	.66044	1.94500	.67691	2.09510	9
10	.62809	1.68884	.64435	1.81175	.66071	1.94737	.67718	2.09774	10
11	.62836	1.69079	.64462	1.81390	.66099	1.94975	.67746	2.10038	11
12	.62863	1.69275	.64489	1.81605	.66126	1.95213	.67773	2.10303	12
13	.62890	1.69471	.64517	1.81821	.66154	1.95452	.67801	2.10568	13
14	.62917	1.69667	.64544	1.82037	.66181	1.95691	.67829	2.10834	14
15	.62944	1.69864	.64571	1.82254	.66208	1.95931	.67856	2.11101	15
16	.62971	1.70061	.64598	1.82471	.66236	1.96171	.67884	2.11367	16
17	.62998	1.70258	.64625	1.82688	.66263	1.96411	.67911	2.11635	17
18	.63025	1.70455	.64653	1.82906	.66290	1.96652	.67939	2.11903	18
19	.63052	1.70653	.64680	1.83124	.66318	1.96893	.67966	2.12171	19
20	.63079	1.70851	.64707	1.83342	.66345	1.97135	.67994	2.12440	20
21	.63106	1.71050	.64734	1.83561	.66373	1.97377	.68021	2.12709	21
22	.63133	1.71249	.64761	1.83780	.66400	1.97619	.68049	2.12979	22
23	.63161	1.71448	.64789	1.83999	.66427	1.97862	.68077	2.13249	23
24	.63188	1.71647	.64816	1.84219	.66455	1.98106	.68104	2.13520	24
25	.63215	1.71847	.64843	1.84439	.66482	1.98349	.68132	2.13791	25
26	.63242	1.72047	.64870	1.84659	.66510	1.98594	.68159	2.14063	26
27	.63269	1.72247	.64898	1.84880	.66537	1.98838	.68187	2.14335	27
28	.63296	1.72448	.64925	1.85102	.66564	1.99083	.68214	2.14608	28
29	.63323	1.72649	.64952	1.85323	.66592	1.99329	.68242	2.14881	29
30	.63350	1.72850	.64979	1.85545	.66619	1.99574	.68270	2.15155	30
31	.63377	1.73052	.65007	1.85767	.66647	1.99821	.68297	2.15429	31
32	.63404	1.73254	.65034	1.85990	.66674	2.00067	.68325	2.15704	32
33	.63431	1.73456	.65061	1.86213	.66702	2.00315	.68352	2.15979	33
34	.63458	1.73659	.65088	1.86437	.66729	2.00562	.68380	2.16255	34
35	.63485	1.73862	.65116	1.86661	.66756	2.00810	.68408	2.16531	35
36	.63512	1.74065	.65143	1.86885	.66784	2.01059	.68435	2.16808	36
37	.63539	1.74269	.65170	1.87109	.66811	2.01308	.68463	2.17085	37
38	.63566	1.74473	.65197	1.87334	.66839	2.01557	.68490	2.17363	38
39	.63594	1.74677	.65225	1.87560	.66866	2.01807	.68518	2.17641	39
40	.63621	1.74881	.65252	1.87785	.66894	2.02057	.68546	2.17920	40
41	.63648	1.75086	.65279	1.88011	.66921	2.02308	.68573	2.18199	41
42	.63675	1.75292	.65306	1.88238	.66949	2.02559	.68601	2.18479	42
43	.63702	1.75497	.65334	1.88465	.66976	2.02810	.68628	2.18759	43
44	.63729	1.75703	.65361	1.88692	.67003	2.03062	.68656	2.19040	44
45	.63756	1.75909	.65388	1.88920	.67031	2.03315	.68684	2.19322	45
46	.63783	1.76116	.65416	1.89148	.67058	2.03568	.68711	2.19604	46
47	.63810	1.76323	.65443	1.89376	.67086	2.03821	.68739	2.19886	47
48	.63838	1.76530	.65470	1.89605	.67113	2.04075	.68767	2.20169	48
49	.63865	1.76737	.65497	1.89834	.67141	2.04329	.68794	2.20453	49
50	.63892	1.76945	.65525	1.90063	.67168	2.04584	.68822	2.20737	50
51	.63919	1.77154	.65552	1.90293	.67196	2.04839	.68849	2.21021	51
52	.63946	1.77362	.65579	1.90524	.67223	2.05094	.68877	2.21306	52
53	.63973	1.77571	.65607	1.90754	.67251	2.05350	.68905	2.21592	53
54	.64000	1.77780	.65634	1.90986	.67278	2.05607	.68932	2.21878	54
55	.64027	1.77990	.65661	1.91217	.67306	2.05864	.68960	2.22165	55
56	.64055	1.78200	.65689	1.91449	.67333	2.06121	.68988	2.22452	56
57	.64082	1.78410	.65716	1.91681	.67361	2.06379	.69015	2.22740	57
58	.64109	1.78621	.65743	1.91914	.67388	2.06637	.69043	2.23028	58
59	.64136	1.78832	.65771	1.92147	.67415	2.06896	.69071	2.23317	59
	.64163	1.79043	.65798	1.92380	.67443	2.07155	.69098	2.23607	60

′	72°		73°		74°		75°	
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.
0	.69098	2.23607	.70763	2.42030	.72436	2.62796	.74118	2.86370
1	.69126	2.23897	.70791	2.42356	.72464	2.63164	.74146	2.86790
2	.69154	2.24187	.70818	2.42683	.72492	2.63533	.74174	2.87211
3	.69181	2.24478	.70846	2.43010	.72520	2.63903	.74202	2.87633
4	.69209	2.24770	.70874	2.43337	.72548	2.64274	.74231	2.88056
5	.69237	2.25062	.70902	2.43666	.72576	2.64645	.74259	2.88479
6	.69264	2.25355	.70930	2.43995	.72604	2.65018	.74287	2.88904
7	.69292	2.25648	.70958	2.44324	.72632	2.65391	.74315	2.89330
8	.69320	2.25942	.70985	2.44655	.72660	2.65765	.74343	2.89756
9	.69347	2.26237	.71013	2.44986	.72688	2.66140	.74371	2.90184
10	.69375	2.26531	.71041	2.45317	.72716	2.66515	.74399	2.90613
11	.69403	2.26827	.71069	2.45650	.72744	2.66892	.74427	2.91042
12	.69430	2.27123	.71097	2.45983	.72772	2.67269	.74455	2.91473
13	.69458	2.27420	.71125	2.46316	.72800	2.67647	.74484	2.91904
14	.69486	2.27717	.71153	2.46651	.72828	2.68025	.74512	2.92337
15	.69514	2.28015	.71180	2.46986	.72856	2.68405	.74540	2.92770
16	.69541	2.28313	.71208	2.47321	.72884	2.68785	.74568	2.93204
17	.69569	2.28612	.71236	2.47658	.72912	2.69167	.74596	2.93640
18	.69597	2.28912	.71264	2.47995	.72940	2.69549	.74624	2.94076
19	.69624	2.29212	.71292	2.48333	.72968	2.69931	.74652	2.94514
20	.69652	2.29512	.71320	2.48671	.72996	2.70315	.74680	2.94952
21	.69680	2.29814	.71348	2.49010	.73024	2.70700	.74709	2.95392
22	.69708	2.30115	.71375	2.49350	.73052	2.71085	.74737	2.95832
23	.69735	2.30418	.71403	2.49691	.73080	2.71471	.74765	2.96274
24	.69763	2.30721	.71431	2.50032	.73108	2.71858	.74793	2.96716
25	.69791	2.31024	.71459	2.50374	.73136	2.72246	.74821	2.97160
26	.69818	2.31328	.71487	2.50716	.73164	2.72635	.74849	2.97604
27	.69846	2.31633	.71515	2.51060	.73192	2.73024	.74878	2.98050
28	.69874	2.31939	.71543	2.51404	.73220	2.73414	.74906	2.98497
29	.69902	2.32244	.71571	2.51748	.73248	2.73806	.74934	2.98944
30	.69929	2.32551	.71598	2.52094	.73276	2.74198	.74962	2.99393
31	.69957	2.32858	.71626	2.52440	.73304	2.74591	.74990	2.99843
32	.69985	2.33166	.71654	2.52787	.73332	2.74984	.75018	3.00293
33	.70013	2.33474	.71682	2.53134	.73360	2.75379	.75047	3.00745
34	.70040	2.33783	.71710	2.53482	.73388	2.75775	.75075	3.01198
35	.70068	2.34092	.71738	2.53831	.73416	2.76171	.75103	3.01652
36	.70096	2.34403	.71766	2.54181	.73444	2.76568	.75131	3.02107
37	.70124	2.34713	.71794	2.54531	.73472	2.76966	.75159	3.02563
38	.70151	2.35025	.71822	2.54883	.73500	2.77365	.75187	3.03020
39	.70179	2.35336	.71850	2.55235	.73529	2.77765	.75216	3.03479
40	.70207	2.35649	.71877	2.55587	.73557	2.78166	.75244	3.03938
41	.70235	2.35962	.71905	2.55940	.73585	2.78568	.75272	3.04398
42	.70263	2.36276	.71933	2.56294	.73613	2.78970	.75300	3.04860
43	.70290	2.36590	.71961	2.56649	.73641	2.79374	.75328	3.05322
44	.70318	2.36905	.71989	2.57005	.73669	2.79778	.75356	3.05786
45	.70346	2.37221	.72017	2.57361	.73697	2.80183	.75385	3.06251
46	.70374	2.37537	.72045	2.57718	.73725	2.80589	.75413	3.06717
47	.70401	2.37854	.72073	2.58076	.73753	2.80996	.75441	3.07184
48	.70429	2.38171	.72101	2.58434	.73781	2.81404	.75469	3.07652
49	.70457	2.38489	.72129	2.58794	.73809	2.81813	.75497	3.08121
50	.70485	2.38808	.72157	2.59154	.73837	2.82223	.75526	3.08591
51	.70513	2.39128	.72185	2.59514	.73865	2.82633	.75554	3.09063
52	.70540	2.39448	.72213	2.59876	.73893	2.83045	.75582	3.09535
53	.70568	2.39768	.72241	2.60238	.73921	2.83457	.75610	3.10009
54	.70596	2.40089	.72269	2.60601	.73950	2.83871	.75639	3.10484
55	.70624	2.40411	.72296	2.60965	.73978	2.84285	.75667	3.10960
56	.70652	2.40734	.72324	2.61330	.74006	2.84700	.75695	3.11437
57	.70679	2.41057	.72352	2.61695	.74034	2.85116	.75723	3.11915
58	.70707	2.41381	.72380	2.62061	.74062	2.85533	.75751	3.12394
59	.70735	2.41705	.72408	2.62428	.74090	2.85951	.75780	3.12875
60	.70763	2.42030	.72436	2.62796	.74118	2.86370	.75808	3.13357

′	76°		77°		78°		79°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.75808	3.13357	.77505	3.44541	.79209	3.80973	.80919	4.24084	0
1	.75836	3.13839	.77533	3.45102	.79237	3.81633	.80948	4.24870	1
2	.75864	3.14323	.77562	3.45664	.79266	3.82294	.80976	4.25658	2
3	.75892	3.14809	.77590	3.46228	.79294	3.82956	.81005	4.26448	3
4	.75921	3.15295	.77618	3.46793	.79323	3.83621	.81033	4.27241	4
5	.75949	3.15782	.77647	3.47360	.79351	3.84288	.81062	4.28036	5
6	.75977	3.16271	.77675	3.47928	.79380	3.84956	.81090	4.28833	6
7	.76005	3.16761	.77703	3.48498	.79408	3.85627	.81119	4.29634	7
8	.76034	3.17252	.77732	3.49069	.79437	3.86299	.81148	4.30436	8
9	.76062	3.17744	.77760	3.49642	.79465	3.86973	.81176	4.31241	9
10	.76090	3.18238	.77788	3.50216	.79493	3.87649	.81205	4.32049	10
11	.76118	3.18733	.77817	3.50791	.79522	3.88327	.81233	4.32859	11
12	.76147	3.19228	.77845	3.51368	.79550	3.89007	.81262	4.33671	12
13	.76175	3.19725	.77874	3.51947	.79579	3.89689	.81290	4.34486	13
14	.76203	3.20224	.77902	3.52527	.79607	3.90373	.81319	4.35304	14
15	.76231	3.20723	.77930	3.53109	.79636	3.91058	.81348	4.36124	15
16	.76260	3.21224	.77959	3.53692	.79664	3.91746	.81376	4.36947	16
17	.76288	3.21726	.77987	3.54277	.79693	3.92436	.81405	4.37772	17
18	.76316	3.22229	.78015	3.54863	.79721	3.93128	.81433	4.38600	18
19	.76344	3.22734	.78044	3.55451	.79750	3.93821	.81462	4.39430	19
20	.76373	3.23239	.78072	3.56041	.79778	3.94517	.81491	4.40263	20
21	.76401	3.23746	.78101	3.56632	.79807	3.95215	.81519	4.41099	21
22	.76429	3.24255	.78129	3.57224	.79835	3.95914	.81548	4.41937	22
23	.76158	3.24764	.78157	3.57819	.79864	3.96616	.81576	4.42778	23
24	.76486	3.25275	.78186	3.58414	.79892	3.97320	.81605	4.43622	24
25	.76514	3.25787	.78214	3.59012	.79921	3.98025	.81633	4.44468	25
26	.76542	3.26300	.78242	3.59611	.79949	3.98733	.81662	4.45317	26
27	.76571	3.26814	.78271	3.60211	.79978	3.99443	.81691	4.46169	27
28	.76599	3.27330	.78299	3.60813	.80006	4.00155	.81719	4.47023	28
29	.76627	3.27847	.78328	3.61417	.80035	4.00869	.81748	4.47881	29
30	.76655	3.28366	.78356	3.62023	.80063	4.01585	.81776	4.48740	30
31	.76684	3.28885	.78384	3.62630	.80092	4.02303	.81805	4.49603	31
32	.76712	3.29406	.78413	3.63238	.80120	4.03024	.81834	4.50468	32
33	.76740	3.29929	.78441	3.63849	.80149	4.03746	.81862	4.51337	33
34	.76769	3.30452	.78470	3.64461	.80177	4.04471	.81891	4.52208	34
35	.76797	3.30977	.78498	3.65074	.80206	4.05197	.81919	4.53081	35
36	.76825	3.31503	.78526	3.65690	.80234	4.05926	.81948	4.53958	36
37	.76854	3.32031	.78555	3.66307	.80263	4.06657	.81977	4.54837	37
38	.76882	3.32560	.78583	3.66925	.80291	4.07390	.82005	4.55720	38
39	.76910	3.33090	.78612	3.67545	.80320	4.08125	.82034	4.56605	39
40	.76938	3.33622	.78640	3.68167	.80348	4.08863	.82063	4.57493	40
41	.76967	3.34154	.78669	3.68791	.80377	4.09602	.82091	4.58383	41
42	.76995	3.34689	.78697	3.69417	.80405	4.10344	.82120	4.59277	42
43	.77023	3.35224	.78725	3.70044	.80434	4.11088	.82148	4.60174	43
44	.77052	3.35761	.78754	3.70673	.80462	4.11835	.82177	4.61073	44
45	.77080	3.36299	.78782	3.71303	.80491	4.12583	.82206	4.61976	45
46	.77108	3.36839	.78811	3.71935	.80520	4.13334	.82234	4.62881	46
47	.77137	3.37380	.78839	3.72569	.80548	4.14087	.82263	4.63790	47
48	.77165	3.37923	.78868	3.73205	.80577	4.14842	.82292	4.64701	48
49	.77193	3.38466	.78896	3.73843	.80605	4.15599	.82320	4.65616	49
50	.77222	3.39012	.78924	3.74482	.80634	4.16359	.82349	4.66533	50
51	.77250	3.39558	.78953	3.75123	.80662	4.17121	.82377	4.67454	51
52	.77278	3.40106	.78981	3.75766	.80691	4.17886	.82406	4.68377	52
53	.77307	3.40656	.79010	3.76411	.80719	4.18652	.82435	4.69304	53
54	.77335	3.41206	.79038	3.77057	.80748	4.19421	.82463	4.70234	54
55	.77363	3.41759	.79067	3.77705	.80776	4.20193	.82492	4.71166	55
56	.77392	3.42312	.79095	3.78355	.80805	4.20966	.82521	4.72102	56
57	.77420	3.42867	.79123	3.79007	.80833	4.21742	.82549	4.73041	57
58	.77148	3.43424	.79152	3.79661	.80862	4.22521	.82578	4.73983	58
59	.77477	3.43982	.79180	3.80316	.80891	4.23301	.82607	4.74929	59
60	.77505	3.44541	.79209	3.80973	.80919	4.24084	.82635	4.75877	60

′	80°		81°		82°		83°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.82635	4.75877	.84357	5.39245	.86083	6.18530	.87813	7.20551	0
1	.82664	4.76829	.84385	5.40422	.86112	6.20020	.87842	7.22500	1
2	.82692	4.77784	.84414	5.41602	.86140	6.21517	.87871	7.24457	2
3	.82721	4.78742	.84443	5.42787	.86169	6.23019	.87900	7.26425	3
4	.82750	4.79703	.84471	5.43977	.86198	6.24529	.87929	7.28402	4
5	.82778	4.80667	.84500	5.45171	.86227	6.26044	.87957	7.30388	5
6	.82807	4.81635	.84529	5.46369	.86256	6.27566	.87986	7.32384	6
7	.82836	4.82606	.84558	5.47572	.86284	6.29095	.88015	7.34390	7
8	.82864	4.83581	.84586	5.48779	.86313	6.30630	.88044	7.36405	8
9	.82893	4.84558	.84615	5.49991	.86342	6.32171	.88073	7.38431	9
10	.82922	4.85539	.84644	5.51208	.86371	6.33719	.88102	7.40466	10
11	.82950	4.86524	.84673	5.52429	.86400	6.35274	.88131	7.42511	11
12	.82979	4.87511	.84701	5.53655	.86428	6.36835	.88160	7.44566	12
13	.83008	4.88502	.84730	5.54886	.86457	6.38403	.88188	7.46632	13
14	.83036	4.89497	.84759	5.56121	.86486	6.39978	.88217	7.48707	14
15	.83065	4.90495	.84788	5.57361	.86515	6.41560	.88246	7.50793	15
16	.83094	4.91496	.84816	5.58606	.86544	6.43148	.88275	7.52889	16
17	.83122	4.92501	.84845	5.59855	.86573	6.44743	.88304	7.54996	17
18	.83151	4.93509	.84874	5.61110	.86601	6.46346	.88333	7.57113	18
19	.83180	4.94521	.84903	5.62369	.86630	6.47955	.88362	7.59241	19
20	.83208	4.95536	.84931	5.63633	.86659	6.49571	.88391	7.61379	20
21	.83237	4.96555	.84960	5.64902	.86688	6.51194	.88420	7.63528	21
22	.83266	4.97577	.84989	5.66176	.86717	6.52825	.88448	7.65688	22
23	.83294	4.98603	.85018	5.67454	.86746	6.54462	.88477	7.67859	23
24	.83323	4.99633	.85046	5.68738	.86774	6.56107	.88506	7.70041	24
25	.83352	5.00666	.85075	5.70027	.86803	6.57759	.88535	7.72234	25
26	.83380	5.01703	.85104	5.71321	.86832	6.59418	.88564	7.74438	26
27	.83409	5.02743	.85133	5.72620	.86861	6.61085	.88593	7.76653	27
28	.83438	5.03787	.85162	5.73924	.86890	6.62759	.88622	7.78880	28
29	.83467	5.04834	.85190	5.75233	.86919	6.64441	.88651	7.81118	29
30	.83495	5.05886	.85219	5.76547	.86947	6.66130	.88680	7.83367	30
31	.83524	5.06941	.85248	5.77866	.86976	6.67826	.88709	7.85628	31
32	.83553	5.08000	.85277	5.79191	.87005	6.69530	.88737	7.87901	32
33	.83581	5.09062	.85305	5.80521	.87034	6.71242	.88766	7.90186	33
34	.83610	5.10129	.85334	5.81856	.87063	6.72962	.88795	7.92482	34
35	.83639	5.11199	.85363	5.83196	.87092	6.74689	.88824	7.94791	35
36	.83667	5.12273	.85392	5.84542	.87120	6.76424	.88853	7.97111	36
37	.83696	5.13350	.85420	5.85893	.87149	6.78167	.88882	7.99444	37
38	.83725	5.14432	.85449	5.87250	.87178	6.79918	.88911	8.01788	38
39	.83754	5.15517	.85478	5.88612	.87207	6.81677	.88940	8.04146	39
40	.83782	5.16607	.85507	5.89979	.87236	6.83443	.88969	8.06515	40
41	.83811	5.17700	.85536	5.91352	.87265	6.85218	.88998	8.08897	41
42	.83840	5.18797	.85564	5.92731	.87294	6.87001	.89027	8.11292	42
43	.83868	5.19898	.85593	5.94115	.87322	6.88792	.89055	8.13699	43
44	.83897	5.21004	.85622	5.95505	.87351	6.90592	.89084	8.16120	44
45	.83926	5.22113	.85651	5.96900	.87380	6.92400	.89113	8.18553	45
46	.83954	5.23226	.85680	5.98301	.87409	6.94216	.89142	8.20999	46
47	.83983	5.24343	.85708	5.99708	.87438	6.96040	.89171	8.23459	47
48	.84012	5.25464	.85737	6.01120	.87467	6.97873	.89200	8.25931	48
49	.84041	5.26590	.85766	6.02538	.87496	6.99714	.89229	8.28417	49
50	.84069	5.27719	.85795	6.03962	.87524	7.01565	.89258	8.30917	50
51	.84098	5.28853	.85823	6.05392	.87553	7.03423	.89287	8.33430	51
52	.84127	5.29991	.85852	6.06828	.87582	7.05291	.89316	8.35957	52
53	.84155	5.31133	.85881	6.08269	.87611	7.07167	.89345	8.38497	53
54	.84184	5.32279	.85910	6.09717	.87640	7.09052	.89374	8.41052	54
55	.84213	5.33429	.85939	6.11171	.87669	7.10946	.89403	8.43620	55
56	.84242	5.34584	.85967	6.12630	.87698	7.12849	.89431	8.46203	56
57	.84270	5.35743	.85996	6.14096	.87726	7.14760	.89460	8.48800	57
58	.84299	5.36906	.86025	6.15568	.87755	7.16681	.89489	8.51411	58
59	.84328	5.38073	.86054	6.17046	.87784	7.18612	.89518	8.54037	59
60	.84357	5.39245	.86083	6.18530	.87813	7.20551	.89547	8.56	

′	84°		85°		86°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.89547	8.56677	.91284	10.47371	.93024	13.33559	0
1	.89576	8.59332	.91313	10.51199	.93053	13.39547	1
2	.89605	8.62002	.91342	10.55052	.93082	13.45586	2
3	.89634	8.64687	.91371	10.58932	.93111	13.51676	3
4	.89663	8.67387	.91400	10.62837	.93140	13.57817	4
5	.89692	8.70103	.91429	10.66769	.93169	13.64011	5
6	.89721	8.72833	.91458	10.70728	.93198	13.70258	6
7	.89750	8.75579	.91487	10.74714	.93227	13.76558	7
8	.89779	8.78341	.91516	10.78727	.93257	13.82913	8
9	.89808	8.81119	.91545	10.82768	.93286	13.89323	9
10	.89836	8.83912	.91574	10.86837	.93315	13.95788	10
11	.89865	8.86722	.91603	10.90934	.93344	14.02310	11
12	.89894	8.89547	.91632	10.95060	.93373	14.08890	12
13	.89923	8.92389	.91661	10.99214	.93402	14.15527	13
14	.89952	8.95248	.91690	11.03397	.93431	14.22223	14
15	.89981	8.98123	.91719	11.07610	.93460	14.28979	15
16	.90010	9.01015	.91748	11.11852	.93489	14.35795	16
17	.90039	9.03923	.91777	11.16125	.93518	14.42672	17
18	.90068	9.06849	.91806	11.20427	.93547	14.49611	18
19	.90097	9.09792	.91835	11.24761	.93576	14.56614	19
20	.90126	9.12752	.91864	11.29125	.93605	14.63679	20
21	.90155	9.15730	.91893	11.33521	.93634	14.70810	21
22	.90184	9.18725	.91922	11.37948	.93663	14.78005	22
23	.90213	9.21739	.91951	11.42408	.93692	14.85268	23
24	.90242	9.24770	.91980	11.46900	.93721	14.92597	24
25	.90271	9.27819	.92009	11.51424	.93750	14.99995	25
26	.90300	9.30887	.92038	11.55982	.93779	15.07462	26
27	.90329	9.33973	.92067	11.60572	.93808	15.14999	27
28	.90358	9.37077	.92096	11.65197	.93837	15.22607	28
29	.90386	9.40201	.92125	11.69856	.93866	15.30287	29
30	.90415	9.43343	.92154	11.74550	.93895	15.38041	30
31	.90444	9.46505	.92183	11.79278	.93924	15.45869	31
32	.90473	9.49685	.92212	11.84042	.93953	15.53772	32
33	.90502	9.52886	.92241	11.88841	.93982	15.61751	33
34	.90531	9.56106	.92270	11.93677	.94011	15.69808	34
35	.90560	9.59346	.92299	11.98549	.94040	15.77944	35
36	.90589	9.62605	.92328	12.03458	.94069	15.86159	36
37	.90618	9.65885	.92357	12.08040	.94098	15.94456	37
38	.90647	9.69186	.92386	12.13388	.94127	16.02835	38
39	.90676	9.72507	.92415	12.18411	.94156	16.11297	39
40	.90705	9.75849	.92444	12.23472	.94186	16.19843	40
41	.90734	9.79212	.92473	12.28572	.94215	16.28476	41
42	.90763	9.82596	.92502	12.33712	.94244	16.37196	42
43	.90792	9.86001	.92531	12.38891	.94273	16.46005	43
44	.90821	9.89428	.92560	12.44112	.94302	16.54903	44
45	.90850	9.92877	.92589	12.49373	.94331	16.63893	45
46	.90879	9.96348	.92618	12.54676	.94360	16.72975	46
47	.90908	9.99841	.92647	12.60021	.94389	16.82152	47
48	.90937	10.03356	.92676	12.65408	.94418	16.91424	48
49	.90966	10.06894	.92705	12.70838	.94447	17.00794	49
50	.90995	10.10455	.92734	12.76312	.94476	17.10262	50
51	.91024	10.14039	.92763	12.81829	.94505	17.19830	51
52	.91053	10.17646	.92792	12.87391	.94534	17.29501	52
53	.91082	10.21277	.92821	12.92999	.94563	17.39274	53
54	.91111	10.24932	.92850	12.98651	.94592	17.49153	54
55	.91140	10.28610	.92879	13.04350	.94621	17.59139	55
56	.91169	10.32313	.92908	13.10096	.94650	17.69233	56
57	.91197	10.36040	.92937	13.15889	.94679	17.79438	57
58	.91226	10.39792	.92966	13.21730	.94708	17.89755	58
	.91255	10.43569	.92995	13.27620	.94737	18.00185	59
	.91284	10.47371	.93024	13.33559	.94766	18.10732	60

′	87°		88°		89°		′
	Vers.	Exsec.	Vers.	Exsec.	Vers.	Exsec.	
0	.94766	18.10732	.96510	27.65371	.98255	56.29869	0
1	.94795	18.21397	.96539	27.89440	.98284	57.26976	1
2	.94825	18.32182	.96568	28.13917	.98313	58.27431	2
3	.94854	18.43088	.96597	28.38812	.98342	59.31411	3
4	.94883	18.54119	.96626	28.64137	.98371	60.39105	4
5	.94912	18.65275	.96655	28.89903	.98400	61.50715	5
6	.94941	18.76560	.96684	29.16120	.98429	62.66460	6
7	.94970	18.87976	.96714	29.42802	.98458	63.86572	7
8	.94999	18.99524	.96743	29.69960	.98487	65.11304	8
9	.95028	19.11208	.96772	29.97607	.98517	66.40927	9
10	.95057	19.23028	.96801	30.25758	.98546	67.75736	10
11	.95086	19.34989	.96830	30.54425	.98575	69.16047	11
12	.95115	19.47093	.96859	30.83623	.98604	70.62285	12
13	.95144	19.59341	.96888	31.13366	.98633	72.14583	13
14	.95173	19.71737	.96917	31.43671	.98662	73.73586	14
15	.95202	19.84283	.96946	31.74554	.98691	75.39655	15
16	.95231	19.96982	.96975	32.06030	.98720	77.13274	16
17	.95260	20.09838	.97004	32.38118	.98749	78.94968	17
18	.95289	20.22852	.97033	32.70835	.98778	80.85315	18
19	.95318	20.36027	.97062	33.04199	.98807	82.84947	19
20	.95347	20.49368	.97092	33.38232	.98836	84.94561	20
21	.95377	20.62876	.97121	33.72952	.98866	87.14924	21
22	.95406	20.76555	.97150	34.08380	.98895	89.46886	22
23	.95435	20.90409	.97179	34.44539	.98924	91.91387	23
24	.95464	21.04440	.97208	34.81452	.98953	94.49471	24
25	.95493	21.18653	.97237	35.19141	.98982	97.22303	25
26	.95522	21.33050	.97266	35.57633	.99011	100.1119	26
27	.95551	21.47635	.97295	35.96953	.99040	103.1757	27
28	.95580	21.62413	.97324	36.37127	.99069	106.4311	28
29	.95609	21.77386	.97353	36.78185	.99098	109.8966	29
30	.95638	21.92559	.97382	37.20155	.99127	113.5930	30
31	.95667	22.07935	.97411	37.63068	.99156	117.5444	31
32	.95696	22.23520	.97440	38.06957	.99186	121.7780	32
33	.95725	22.39316	.97470	38.51855	.99215	126.3253	33
34	.95754	22.55329	.97499	38.97797	.99244	131.2223	34
35	.95783	22.71563	.97528	39.44820	.99273	136.5111	35
36	.95812	22.88022	.97557	39.92963	.99302	142.2406	36
37	.95842	23.04712	.97586	40.42266	.99331	148.4684	37
38	.95871	23.21637	.97615	40.92772	.99360	155.2623	38
39	.95900	23.38802	.97644	41.44525	.99389	162.7033	39
40	.95929	23.56212	.97673	41.97571	.99418	170.8883	40
41	.95958	23.73873	.97702	42.51961	.99447	179.9350	41
42	.95987	23.91790	.97731	43.07746	.99476	189.9868	42
43	.96016	24.09969	.97760	43.64980	.99505	201.2212	43
44	.96045	24.28414	.97789	44.23720	.99535	213.8600	44
45	.96074	24.47134	.97819	44.84026	.99564	228.1839	45
46	.96103	24.66132	.97848	45.45963	.99593	244.5540	46
47	.96132	24.85417	.97877	46.09596	.99622	263.4427	47
48	.96161	25.04994	.97906	46.74997	.99651	285.4795	48
49	.96190	25.24869	.97935	47.42241	.99680	311.5230	49
50	.96219	25.45051	.97964	48.11406	.99709	342.7752	50
51	.96248	25.65546	.97993	48.82576	.99738	380.9723	51
52	.96277	25.86360	.98022	49.55840	.99767	428.7187	52
53	.96307	26.07503	.98051	50.31290	.99796	490.1070	53
54	.96336	26.28981	.98080	51.09027	.99825	571.9581	54
55	.96365	26.50804	.98109	51.89156	.99855	686.5496	55
56	.96394	26.72978	.98138	52.71790	.99884	858.4369	56
57	.96423	26.95513	.98168	53.57046	.99913	1144.916	57
58	.96452	27.18417	.98197	54.45053	.99942	1717.874	58
59	.96481	27.41700	.98226	55.35946	.99971	3436.747	59
60	.96510	27.65371	.98255	56.29869	1.00000	Infinite	60

Ratio of circumference to diameter	π	3.1415926536
Reciprocal of same	$\frac{1}{\pi}$	.3183098862
	$\sqrt{\pi}$	1.7724538509
	π^2	9.8696044011
Degrees in arc equal to radius	$\frac{180}{\pi}$	57.295779513
Minutes in arc equal to radius	$\frac{10800}{\pi}$	3437.74677078
Seconds in arc equal to radius	$\frac{648000}{\pi}$	206264.806247
Length of seconds pendulum at New York in feet		3.25938
Square root of same		1.8054
Acceleration due to gravity at New York	g	32.1688
Square root of same	$\sqrt{g}$	5.67175
Cubic inches in U. S. gallon		231
Cubic inches in Imperial gallon		277.274
Cubic inches in U. S. bushel		2150.42
Cubic feet in U. S. bushel		1.244456
U. S. gallons in one cubic foot		7.4805
Imperial gallons in one cubic foot		6.2321
Maximum weight of one cubic foot of water (⅝ 100 nearly)		62.4
Number of grains in one pound avoirdupois		7000
Number of grains in one ounce avoirdupois		437.5
Number of grains in one pound troy		5760
Number of grains in one ounce troy		480
Circumference of circle (radius r)		$2\pi r$
Area of circle (radius r)		πr^2
Area of sector (arc of a degrees)		$\frac{a}{360}\pi r^2$
Area of sector (length of arc l)		$\frac{1}{2}lr$
Area of segment (chord $= c$, middle ordinate $= m$), nearly		$\frac{2}{3}cm$
Surface of a sphere (diameter d)		πd^2
Volume of a sphere (diameter d)		$\frac{1}{6}\pi d^3$
Area of a triangle each side unity		.4330
Area of a pentagon each side unity		1.7205
Area of a hexagon each side unity $= .4330 \times 6 =$		2.5980
Area of an octagon each side unity		4.8284

Volume spherical segment of one base of radius r and altitude a is

$$= \tfrac{1}{2}\pi ar^2 + \tfrac{1}{6}\pi a^3 = 1.5708ar^2 + .5236a^3;$$

$$= 1.58ar^2 + .5a^3, \text{ very nearly;}$$

$$= 1.6ar^2 + .5a^3, \text{ nearly.}$$

Let A and a represent the bases of a frustum of a pyramid, L and l the lengths of corresponding sides, h the altitude and a_1 the area of either base for side equal to unity.

Then volume $= \frac{ha_1}{3}(L^2 + l^2 + lL)$.

This formula is *immensely* shorter and better than that given in the books, namely, vol $= \frac{h}{3}(A + a + \sqrt{Aa})$, since this latter requires extra and needless computation in each term, besides the extraction of the square root.

When $l = 0$, we get volume of pyramid $= \frac{ha_1L^2}{3}$.

Let R and r represent the radii of a frustum of a cone, and h its altitude.
Then volume $= \frac{1}{3}\pi h(R^2 + r^2 + rR)$.
If $r = 0$ we have, volume of cone $= \frac{1}{3}h_1\pi R^2$.

CONVERSION OF ENGLISH FEET INTO METRES.

Feet.	0	1	2	3	4	5	6	7	8	9
	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.
0	0.000	0.3048	0.6096	0.9144	1.2192	1.5239	1.8287	2.1335	2.4383	2.7431
10	3.0479	3.3527	3.6575	3.9623	4.2671	4.5719	4.8767	5.1815	5.4863	5.7911
20	6.0959	6.4006	6.7055	7.0102	7.3150	7.6198	7.9246	8.2294	8.5342	8.8390
30	9.1438	9.4486	9.7534	10.058	10.363	10.668	10.972	11.277	11.582	11.887
40	12.192	12.496	12.801	13.106	13.411	13.716	14.020	14.325	14.630	14.935
50	15.239	15.544	15.849	16.154	16.459	16.763	17.068	17.373	17.678	17.983
60	18.287	18.592	18.897	19.202	19.507	19.811	20.116	20.421	20.726	21.031
70	21.335	21.640	21.945	22.250	22.555	22.859	23.164	23.469	23.774	24.079
80	24.383	24.688	24.993	25.298	25.602	25.907	26.212	26.517	26.822	27.126
90	27.431	27.736	28.041	28.346	28.651	28.955	29.260	29.565	29.870	30.174
100	30.479	30.784	31.089	31.394	31.698	32.003	32.308	32.613	32.918	33.222

CONVERSION OF METRES INTO ENGLISH FEET.

Met.	0	1	2	3	4	5	6	7	8	9
	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0	0.000	3.2809	6.5618	9.8427	13.123	16.404	19.685	22.966	26.247	29.528
10	32.809	36.090	39.371	42.651	45.932	49.213	52.494	55.775	59.056	62.337
20	65.618	68.899	72.179	75.461	78.741	82.022	85.303	88.584	91.865	95.146
30	98.427	101.71	104.99	108.27	111.55	114.83	118.11	121.39	124.67	127.96
40	131.24	134.52	137.80	141.08	144.36	147.64	150.92	154.20	157.48	160.76
50	164.04	167.33	170 61	173.89	177.17	180.45	183.73	187.01	190.29	193.57
60	196.85	200.13	203.42	206.70	209.98	213.26	216.54	219.82	223.10	226.38
70	229.66	232.94	236.22	239.51	242.79	246.07	249.35	252.63	255.91	259.19
80	262.47	265.75	269.03	272.31	275.60	278.88	282.16	285.44	288.72	292.00
90	295.28	298.56	391.84	305.12	308.40	311.69	314.97	318.25	321.53	324.81
100	328.09	331.37	334.65	337.93	341.21	344.49	347.78	351.06	354.34	357.62

ONVERSION OF ENGLISH STATUTE-MILES INTO KILOMETRES.

iles.	0	1	2	3	4	5	6	7	8	9
	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.
0	0.0000	1.6093	3.2186	4.8279	6.4372	8.0465	9.6558	11.2652	12.8745	14.4838
10	16.093	17.702	19.312	20.921	22.530	24.139	25.749	27.358	28.967	30.577
20	32.186	33.795	35.405	37.014	38.623	40.232	41.842	43.451	45.060	46.670
30	48.279	49.888	51.498	53.107	54.716	56.325	57.935	59.544	61.153	62.763
40	64.372	65.981	67.591	69.200	70.809	72.418	74.028	75.637	77.246	78.856
50	80.465	82.074	83.684	85.293	86.902	88.511	90.121	91 730	93.339	94.949
60	96.558	98.167	99.777	101.39	102.99	104.60	106.21	107.82	109.43	111.04
70	112.65	114.26	115.87	117.48	119.08	120.69	122.30	123.91	125.52	127.13
80	128.74	130.35	131.96	133.57	135.17	136.78	138.39	140.00	141.61	143.22
90	144.85	146.44	148.05	149.66	151.26	152.87	154.48	156.09	157.70	159.31
100	160.93	162.53	164 14	165 75	167.35	168.96	170.57	172.18	173.79	175.40

ONVERSION OF KILOMETRES INTO ENGLISH STATUTE-MILES

ilom.	0	1	2	3	4	5	6	7	8	9
	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.
0	0.0000	0.6214	1.2427	1.8641	2.4855	3.1069	3.7282	4.3497	4.9711	5.5924
10	6.2138	6.8352	7.4565	8.0780	8.6994	9.3208	9.9421	10.562	11.185	11.805
20	12.427	13.049	13.670	14.292	14.913	15.534	16.156	16.776	17.399	18.019
30	18.641	19.263	19.884	20.506	21.127	21.748	23.370	22.990	23.613	24.233
40	24.855	25.477	26.098	26.720	27.341	27.962	28.584	29.204	29.827	30.447
50	31.069	31.690	32.311	32.933	33.554	34.175	34.797	35.417	36.040	36.660
60	37.282	37.904	38.525	39.147	39.768	40.389	41.011	41.631	42.254	42.874
70	43.497	44.118	44.739	45.361	45.982	46.603	47.225	47.845	48.468	49.088
80	49.711	50.332	50.953	51 575	52.196	52.817	53.439	54.059	54.682	55.302
90	55.924	56.545	57.166	37.788	58.409	59 030	59.652	30.272	60.895	61.515
100	62.138	62.759	63.380	64.002	64.623	35.244	65.866	66.486	67.109	67.729

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